Discrete Mathematics for Part I CST 2019/20
Sets Exercises

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- Suggested supervision schedule
  - On sets, relations, and partial functions: Basic (§§ 1.1, 2.1, 3.1) and core (§§ 1.2, 2.2, 3.2) exercises.
    Lectures 12–14 onwards.
  - On functions, bijections, and equivalence relations: Basic (§§ 4.1, 5.1, 6.1) and core (§§ 4.2, 5.2, 6.2) exercises.
    Lecture 16 onwards.
  - On surjections, injections, and images: Basic (§§ 7.1, 8.1, 9.1) and core (§§ 7.2, 8.2, 9.2) exercises.
    Lecture 17 onwards.
  - On countability: Basic (§ 10.1) and core (§ 10.2) exercises.
    Lecture 18 onwards.
- Suggested Easter-break work
  - 2019 Paper 2 Question 9
  - 2018 Paper 2 Questions 7 (b), 8 (b) & (c), and 9 (b) & (c)
  - 2017 Paper 2 Questions 7 (b), 8 (b) & (c), and 9 (b) & (c)
  - 2016 Paper 2 Question 9 (b) & (c)
  - 2015 Paper 2 Questions 7 (c), 8 (c), and 9 (b) & (c)
  - 2014 Paper 2 Question 8
  - 2013 Paper 2 Question 5
  - 2011 Paper 2 Question 5
  - 2009 Paper 1 Question 4
  - 2008 Paper 2 Question 3
  - 2007 Paper 2 Question 5
  - 2006 Paper 2 Question 5

1 On sets

1.1 Basic exercises

1. Prove the following statements:
   (a) Reflexivity: $\forall\ \mathsf{sets}\ A.\, A \subseteq A$.
   (b) Transitivity: $\forall\ \mathsf{sets}\ A, B, C.\, (A \subseteq B \ \wedge\ B \subseteq C) \implies A \subseteq C$.
   (c) Antisymmetry: $\forall\ \mathsf{sets}\ A, B.\, (A \subseteq B \ \wedge\ B \subseteq A) \iff A = B$.
2. Prove the following statements:

(a) $\forall \text{ set } S.\, \emptyset \subseteq S$.
(b) $\forall \text{ set } S.\, (\forall x.\, x \notin S) \iff S = \emptyset$.

3. Find the union and intersection of:

   (a) $\{1,2,3,4,5\}$ and $\{-1,1,3,5,7\}$;
   (b) $\{x \in \mathbb{R} \mid x > 7\}$ and $\{x \in \mathbb{N} \mid x > 5\}$.

4. Find the product of $\{1,2,3,4,5\}$ and $\{-1,1,3,5,7\}$.

5. Let $I = \{2,3,4,5\}$, and for each $i \in I$ let $A_i = \{i, i+1, i-1, 2 \cdot i\}$.

   (a) List the elements of all the sets A_i for $i \in I$.
   (b) Let $\{\, A_i \mid i \in I \,\}$ stand for $\{A_2, A_3, A_4, A_5\}$. Find $\bigcup \{A_i \mid i \in I\}$ and $\bigcap \{A_i \mid i \in I\}$.

6. Find the disjoint union of $\{1,2,3,4,5\}$ and $\{-1,1,3,5,7\}$.

7. Let U be a set. For all $A, B \in \mathcal{P}(U)$ prove that

   (a) $A^{\mathrm{c}} = B \iff (A \cup B = U \,\wedge\, A \cap B = \emptyset)$,
   (b) $(A^{\mathrm{c}})^{\mathrm{c}} = A$, and
   (c) the De Morgan's laws:
   $$(A \cup B)^{\mathrm{c}} = A^{\mathrm{c}} \cap B^{\mathrm{c}} \text{ and } (A \cap B)^{\mathrm{c}} = A^{\mathrm{c}} \cup B^{\mathrm{c}} \ .$$

8. Establish the laws of the powerset Boolean algebra.

1.2 Core exercises

1. Either prove or disprove that, for all sets A and B,

   (a) $A \subseteq B \implies \mathcal{P}(A) \subseteq \mathcal{P}(B)$,
   (b) $\mathcal{P}(A \cup B) \subseteq \mathcal{P}(A) \cup \mathcal{P}(B)$,
   (c) $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$.
   (d) $\mathcal{P}(A \cap B) \subseteq \mathcal{P}(A) \cap \mathcal{P}(B)$,
   (e) $\mathcal{P}(A) \cap \mathcal{P}(B) \subseteq \mathcal{P}(A \cap B)$.

2. Let U be a set. For all $A, B \in \mathcal{P}(U)$ prove that the following statements are equivalent.

   (a) $A \cup B = B$.
   (b) $A \subseteq B$.
   (c) $A \cap B = A$.
   (d) $B^{\mathrm{c}} \subseteq A^{\mathrm{c}}$.

3. For sets A, B, C, D, either prove or disprove the following statements.

   (a) $(A \subseteq B \,\wedge\, C \subseteq D) \implies A \times C \subseteq B \times D$.
   (b) $(A \cup C) \times (B \cup D) \subseteq (A \times B) \cup (C \times D)$.
   (c) $(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D)$.
   (d) $A \times (B \cup D) \subseteq (A \times B) \cup (A \times D)$.
   (e) $(A \times B) \cup (A \times D) \subseteq A \times (B \cup D)$.

4. Prove or disprove the following statements for all sets A, B, C, D:

(a) $(A \subseteq B \ \land \ C \subseteq D) \implies A \uplus C \subseteq B \uplus D$,
(b) $(A \cup B) \uplus C \subseteq (A \uplus C) \cup (B \uplus C)$,
(c) $(A \uplus C) \cup (B \uplus C) \subseteq (A \cup B) \uplus C$,
(d) $(A \cap B) \uplus C \subseteq (A \uplus C) \cap (B \uplus C)$,
(e) $(A \uplus C) \cap (B \uplus C) \subseteq (A \cap B) \uplus C$.

5. For $\mathcal{F} \subseteq \mathcal{P}(A)$, let $\mathcal{U} = \big\{ X \subseteq A \mid \forall S \in \mathcal{F}.\ S \subseteq X \big\} \subseteq \mathcal{P}(A)$. Prove that $\bigcup \mathcal{F} = \bigcap \mathcal{U}$.
Analogously, define $\mathcal{L} \subseteq \mathcal{P}(A)$ such that $\bigcap \mathcal{F} = \bigcup \mathcal{L}$. Also prove this statement.

6. Prove that, for all collections of sets $\mathcal{F}$, it holds that
$$\forall \text{ set } U.\ \bigcup \mathcal{F} \subseteq U \iff \big(\forall X \in \mathcal{F}.\, X \subseteq U\big) \quad .$$

1.3 Optional advanced exercises

Prove that for all collections of sets $\mathcal{F}_1$ and $\mathcal{F}_2$,
$$\big(\bigcup \mathcal{F}_1\big) \cup \big(\bigcup \mathcal{F}_2\big) = \bigcup (\mathcal{F}_1 \cup \mathcal{F}_2) \quad .$$
State and prove the analogous property for intersections of non-empty collections of sets.

2 On relations

2.1 Basic exercises

1. Let $A = \{1, 2, 3, 4\}$, $B = \{a, b, c, d\}$, and $C = \{x, y, z\}$.
Let $R = \big\{(1, a), (2, d), (3, a), (3, b), (3, d)\big\} : A \nrightarrow B$ and $S = \big\{(b, x), (b, x), (c, y), (d, z)\big\} : B \nrightarrow C$.
What is the composition $S \circ R : A \nrightarrow C$?

2. Prove that relational composition is associative and has the identity relation as neutral element.

3. For a relation $R : A \nrightarrow B$, let its *opposite*, or *dual*, $R^{\mathrm{op}} : B \nrightarrow A$ be defined by
$$b\, R^{\mathrm{op}}\, a \iff a\, R\, b \quad .$$
For $R, S : A \nrightarrow B$, prove that
(a) $R \subseteq S \implies R^{\mathrm{op}} \subseteq S^{\mathrm{op}}$.
(b) $(R \cap S)^{\mathrm{op}} = R^{\mathrm{op}} \cap S^{\mathrm{op}}$.
(c) $(R \cup S)^{\mathrm{op}} = R^{\mathrm{op}} \cup S^{\mathrm{op}}$.

4. For a relation R on a set A, prove that R is antisymmetric iff $R \cap R^{\mathrm{op}} \subseteq \mathrm{id}_A$.

2.2 Core exercises

1. Let $\mathcal{F} \subseteq \mathcal{P}(A \times B)$ be a collection of relations from A to B. Prove that,
(a) for all $R : X \nrightarrow A$,
$$\big(\bigcup \mathcal{F}\big) \circ R \ = \ \bigcup \big\{ S \circ R \mid S \in \mathcal{F} \big\} \ : X \nrightarrow B \quad ,$$
and that,
(b) for all $R : B \nrightarrow Y$,

$$R \circ \left(\bigcup \mathcal{F}\right) \;=\; \bigcup \left\{\, R \circ S \mid S \in \mathcal{F} \,\right\} \;: A \mathrel{+\!\!\!\rightarrow} Y \quad .$$

What happens in the case of big intersections?

2. For a relation R on a set A, let

$$\mathcal{T}_R \;=\; \left\{\, Q \subseteq A \times A \;\mid\; R \subseteq Q \,\wedge\, Q \text{ is transitive} \,\right\} \quad .$$

For $R^{\circ+} = R \circ R^{\circ*}$, prove that (i) $R^{\circ+} \in \mathcal{T}_R$ and (ii) $R^{\circ+} \subseteq \bigcap \mathcal{T}_R$. Hence, $R^{\circ+} = \bigcap \mathcal{T}_R$.

3 On partial functions

3.1 Basic exercises

1. Let $A_2 = \{1, 2\}$ and $A_3 = \{a, b, c\}$. List the elements of the four sets $(A_i \rightharpoonup A_j)$ for $i, j \in \{2, 3\}$.

2. Prove that a relation $R : A \mathrel{+\!\!\!\rightarrow} B$ is a partial function iff $R \circ R^{\mathrm{op}} \subseteq \mathrm{id}_B$.

3. Prove that the identity relation is a partial function, and that the composition of partial functions yields a partial function.

3.2 Core exercises

1. Show that $\big(\mathrm{PFun}(A, B), \subseteq\big)$ is a partial order.

2. Show that the intersection of a non-empty collection of partial functions in $\mathrm{PFun}(A, B)$ is a partial function in $\mathrm{PFun}(A, B)$.

3. Show that the union of two partial functions in $\mathrm{PFun}(A, B)$ is a relation that need not be a partial function; but that for $f, g \in \mathrm{PFun}(A, B)$ such that $f \subseteq h \supseteq g$ for some $h \in \mathrm{PFun}(A, B)$, the union $f \cup g$ is a partial function in $\mathrm{PFun}(A, B)$.

4 On functions

4.1 Basic exercises

1. Let $A_2 = \{1, 2\}$ and $A_3 = \{a, b, c\}$. List the elements of the four sets $(A_i \Rightarrow A_j)$ for $i, j \in \{2, 3\}$.

2. A relation $R : A \mathrel{+\!\!\!\rightarrow} B$ is said to be total whenever $\forall\, a \in A.\, \exists\, b \in B.\, a \mathrel{R} b$. Prove that this is equivalent to $\mathrm{id}_A \subseteq R^{\mathrm{op}} \circ R$.

   Conclude that a relation $R : A \mathrel{+\!\!\!\rightarrow} B$ is a function iff $R \circ R^{\mathrm{op}} \subseteq \mathrm{id}_B$ and $\mathrm{id}_A \subseteq R^{\mathrm{op}} \circ R$.

3. Prove that the identity partial function is a function, and that the composition of functions yields a function.

4.2 Core exercises

1. Find endofunctions $f, g : A \to A$ such that $f \circ g \neq g \circ f$. Prove your claim.

2. Let $\chi : \mathcal{P}(U) \to (U \Rightarrow [2])$ be the function mapping subsets S of U to their characteristic (or indicator) functions $\chi_S : U \to [2]$.

   (a) Prove that, for all $x \in U$,

   - $\chi_{A \cup B}(x) \;=\; \big(\chi_A(x) \text{ OR } \chi_B(x)\big) \;=\; \max\big(\chi_A(x), \chi_B(x)\big)$,
   - $\chi_{A \cap B}(x) \;=\; \big(\chi_A(x) \text{ AND } \chi_B(x)\big) \;=\; \min\big(\chi_A(x), \chi_B(x)\big)$,

- $\chi_{A^c}(x) = \text{NOT}\big(\chi_A(x)\big) = \big(1 - \chi_A(x)\big)$.

(b) For what construction $A?B$ on sets A and B it holds that

$$\chi_{A?B}(x) = \big(\chi_A(x) \text{ XOR } \chi_B(x)\big) = \big(\chi_A(x) +_2 \chi_B(x)\big)$$

for all $x \in U$? Prove your claim.

4.3 Optional advanced exercises

Consider a set A together with an element $a \in A$ and an endofunction $f : A \to A$.

Say that a relation $R \subseteq \mathbb{N} \times A$ is (a, f)-*closed* whenever

$$(0, a) \in R \quad \text{and} \quad \forall (n, x) \in \mathbb{N} \times A.\, (n, x) \in R \implies \big(n+1, f(x)\big) \in R\ .$$

Define the relation $F \subseteq \mathbb{N} \times A$ as

$$F = \bigcap \{\, R \subseteq \mathbb{N} \times A \mid R \text{ is } (a, f)\text{-closed} \,\}\ .$$

(a) Prove that the relation F is (a, f)-closed.

(b) Prove that the relation F is total; that is, $\forall\, n \in \mathbb{N}.\, \exists\, y \in A.\, (n, y) \in F$.

(c) Prove that the relation F is a (total) function $\mathbb{N} \to A$; that is,

$$\forall\, n \in \mathbb{N}.\, \exists!\, y \in A.\, (n, y) \in F\ .$$

Hint: Proceed by induction. Observe that, in view of the previous item, to show that $\exists!\, y \in A.\, (\ell, y) \in F$ it suffices to exhibit an (a, f)-closed relation R_ℓ such that $\exists!\, y \in A.\, (\ell, y) \in R_\ell$. (Why?) For instance, as the relation $R_0 = \{\, (m, y) \in \mathbb{N} \times A \mid m = 0 \implies y = a \,\}$ is (a, f)-closed one has that $(0, y) \in F \implies (0, y) \in R_0 \implies y = a$.

(d) Show that if h is a function $\mathbb{N} \to A$ such that $h(0) = a$ and $\forall\, n \in \mathbb{N}.\, h(n+1) = f\big(h(n)\big)$ then $h = F$.

Thus, for every set A together with an element $a \in A$ and an endofunction $f : A \to A$ there exists a unique function $F : \mathbb{N} \to A$, typically said to be *inductively defined*, satisfying the recurrence relation

$$F(n) = \begin{cases} a & \text{, for } n = 0 \\ f\big(F(n-1)\big) & \text{, for } n \geq 1 \end{cases}$$

5 On bijections

5.1 Basic exercises

1. (a) Give examples of functions that have (i) none, (ii) exactly one, and (iii) more than one retraction.
   (b) Give examples of functions that have (i) none, (ii) exactly one, and (iii) more than one section.

2. Let n be an integer.
   (a) How many sections are there for the absolute-value map $[-n..n] \to [0..n] : x \mapsto |\,x\,|$?
   (b) How many retractions are there for the exponential map $[0..n] \to [0..2^n] : x \mapsto 2^x$?

3. Give an example of two sets A and B and a function $f : A \to B$ satisfying both:
   (i) there is a retraction for f, and
   (ii) there is no section for f.

   Explain how you know that f has these two properties.

4. Prove that the identity function is a bijection, and that the composition of bijections yields a bijection.

5. For $f : A \to B$, prove that if there are $g, h : B \to A$ such that $g \circ f = \mathrm{id}_A$ and $f \circ h = \mathrm{id}_B$ then $g = h$.
   Conclude as a corollary that, whenever it exists, the inverse of a function is unique.

5.2 Core exercises

1. We say that two functions $s : A \to B$ and $r : B \to A$ are a *section-retraction* pair whenever $r \circ s = \mathrm{id}_A$; and that a function $e : B \to B$ is an *idempotent* whenever $e \circ e = e$.

   (a) Show that if $s : A \to B$ and $r : B \to A$ are a section-retraction pair then the composite $s \circ r : B \to B$ is an idempotent.

   (b) Prove that for every idempotent $e : B \to B$ there exists a set A and a section-retraction pair $s : A \to B$ and $r : B \to A$ such that $s \circ r = e$.

   (c) Let $p : C \to D$ and $q : D \to C$ be functions such that $p \circ q \circ p = p$. Can you conclude that

   - $p \circ q$ is idempotent? If so, how?
   - $q \circ p$ is idempotent? If so, how?

2. Prove the isomorphisms of the *Calculus of Bijections, I.*

3. Prove that, for all $m, n \in \mathbb{N}$,

   (a) $\mathcal{P}\big([n]\big) \cong [2^n]$

   (b) $[m] \times [n] \cong [m \cdot n]$

   (c) $[m] \uplus [n] \cong [m + n]$

   (d) $\big([m] \rightharpoonup [n]\big) \cong [(n+1)^m]$

   (e) $\big([m] \Rightarrow [n]\big) \cong [n^m]$

   (f) $\mathrm{Bij}\big([n], [n]\big) \cong [n!]$

6 On equivalence relations

6.1 Basic exercises

1. For a relation R on a set A, prove that

   - R is reflexive iff $\mathrm{id}_A \subseteq R$,
   - R is symmetric iff $R \subseteq R^{\mathrm{op}}$,
   - R is transitive iff $R \circ R \subseteq R$.

2. Prove that the isomorphism relation $\cong$ between sets is an equivalence relation.

3. Prove that the identity relation id_A on a set A is an equivalence relation and that $A/_{\mathrm{id}_A} \cong A$.

4. Show that, for a positive integer m, the relation $\equiv_m$ on $\mathbb{Z}$ given by

   $$x \equiv_m y \iff x \equiv y \pmod{m} \quad .$$

   is an equivalence relation.

5. Show that the relation $\equiv$ on $\mathbb{Z} \times \mathbb{N}^+$ given by

   $$(a, b) \equiv (x, y) \iff a \cdot y = x \cdot b$$

   is an equivalence relation.

6. Let B be a subset of a set A. Define the relation E on $\mathcal{P}(A)$ by

   $$\big(X, Y\big) \in E \iff X \cap B = Y \cap B \quad .$$

   Show that E is an equivalence relation.

6.2 Core exercises

1. Let E_1 and E_2 be two equivalence relations on a set A. Either prove or disprove the following statements.

   (a) $E_1 \cup E_2$ is an equivalence relation on A.

   (b) $E_1 \cap E_2$ is an equivalence relation on A.

2. For an equivalence relation E on a set A, show that $[a_1]_E = [a_2]_E$ iff $a_1 \mathrel{E} a_2$, where $[a]_E = \{\, x \in A \mid x \mathrel{E} a \,\}$.

3. For a function $f : A \to B$ define a relation $\equiv_f$ on A by the rule

$$a \equiv_f a' \iff f(a) = f(a')$$

   for all $a, a' \in A$.

   (a) Show that for every function $f : A \to B$, the relation $\equiv_f$ is an equivalence on A.

   (b) Prove that every equivalence relation E on a set A is equal to $\equiv_q$ for q the quotient function $A \twoheadrightarrow A/_E : a \mapsto [a]_E$.

   (c) Prove that for every surjection $f : A \twoheadrightarrow B$,

$$B \;\cong\; \left(A/_{\equiv_f}\right) \quad .$$

7 On surjections

7.1 Basic exercises

1. Give three examples of functions that are surjective and three examples of functions that are not.

2. Prove that the identity function is a surjection, and that the composition of surjections yields a surjection.

7.2 Core exercises

From surjections $A \twoheadrightarrow B$ and $X \twoheadrightarrow Y$ define, and prove surjective, functions $A \times X \twoheadrightarrow B \times Y$ and $A \uplus X \twoheadrightarrow B \uplus Y$.

8 On injections

8.1 Basic exercises

1. Give three examples of functions that are injective and three of functions that are not.

2. Prove that the identity function is an injection, and that the composition of injections yields an injection.

8.2 Core exercises

From injections $A \rightarrowtail B$ and $X \rightarrowtail Y$ define, and prove injective, functions $A \times X \rightarrowtail B \times Y$ and $A \uplus X \rightarrowtail B \uplus Y$.

9 On images

9.1 Basic exercises

1. What is the direct image of $\mathbb{N}$ under the integer square-root relation $R_2 = \{\, (m,n) \mid m = n^2 \,\} : \mathbb{N} \nrightarrow \mathbb{Z}$? And the inverse image of $\mathbb{N}$?

2. For a relation $R : A \nrightarrow B$, show that

   (a) $\overrightarrow{R}(X) \;=\; \bigcup_{x\in X} \overrightarrow{R}\big(\{x\}\big)$ for all $X \subseteq A$, and

   (b) $\overleftarrow{R}(Y) \;=\; \big\{\, a \in A \mid \overrightarrow{R}\big(\{a\}\big) \subseteq Y \,\big\}$ for all $Y \subseteq B$.

9.2 Core exercises

1. For $X \subseteq A$, prove that the direct image $\overrightarrow{f}(X) \subseteq B$ under an injective function $f : A \rightarrowtail B$ is in bijection with X; that is, $X \cong \overrightarrow{f}(X)$.

2. Prove that for a surjective function $f : A \twoheadrightarrow B$, the direct image function $\overrightarrow{f} : \mathcal{P}(A) \to \mathcal{P}(B)$ is surjective.

3. Show that, by inverse image,

   $$\text{every map } A \to B \text{ induces a Boolean algebra map } \mathcal{P}(B) \to \mathcal{P}(A)\ .$$

   That is, for every function $f : A \to B$,

   - $\overleftarrow{f}(\emptyset) = \emptyset$
   - $\overleftarrow{f}(X \cup Y) = \overleftarrow{f}(X) \cup \overleftarrow{f}(Y)$
   - $\overleftarrow{f}(B) = A$
   - $\overleftarrow{f}(X \cap Y) = \overleftarrow{f}(X) \cap \overleftarrow{f}(Y)$
   - $\overleftarrow{f}\big(X^{\mathrm{c}}\big) = \big(\overleftarrow{f}(X)\big)^{\mathrm{c}}$

   for all $X, Y \subseteq B$.

9.3 Optional advanced exercises

For a relation $R : A \nrightarrow B$, prove that

(a) $\overrightarrow{R}\big(\bigcup \mathcal{F}\big) \;=\; \bigcup\big\{\, \overrightarrow{R}(X) \mid X \in \mathcal{F} \,\big\} \;\in \mathcal{P}(B)$ for all $\mathcal{F} \in \mathcal{P}(\mathcal{P}(A))$, and

(b) $\overleftarrow{R}\big(\bigcap \mathcal{G}\big) \;=\; \bigcap\big\{\overleftarrow{R}(Y) \mid Y \in \mathcal{G}\big\} \;\in \mathcal{P}(A)$ for all $\mathcal{G} \in \mathcal{P}(\mathcal{P}(B))$.

10 On countability

10.1 Basic exercises

Prove that:

(a) Every finite set is countable.

(b) $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ are countable sets.

10.2 Core exercises

1. For an infinite set S, prove that if there is a surjection $\mathbb{N} \to S$ then there is a bijection $\mathbb{N} \to S$.
2. Prove that:
   (a) Every subset of a countable set is countable.
   (b) The product and disjoint union of countable sets is countable.
3. For an infinite set S, prove that the following are equivalent:
   (a) There is a bijection $\mathbb{N} \to S$.
   (b) There is an injection $S \to \mathbb{N}$.
   (c) There is a surjection $\mathbb{N} \to S$
4. For a set X, prove that there is no injection $\mathcal{P}(X) \to X$.

10.3 Optional advance exercises

Prove that if X and A are countable sets then so are A^*, $\mathcal{P}_{\text{fin}}(A)$, and $(X \rightharpoonup_{\text{fin}} A)$.

11 On indexed sets

11.1 Optional advanced exercises

Prove the isomorphisms of the *Calculus of Bijections, II*.