Calculus of bijections

- $A \cong A$, $A \cong B \implies B \cong A$, $(A \cong B \wedge B \cong C) \implies A \cong C$
- If $A \cong X$ and $B \cong Y$ then

$$\mathcal{P}(A) \cong \mathcal{P}(X) \quad , \quad A \times B \cong X \times Y \quad , \quad A \uplus B \cong X \uplus Y \quad ,$$

$$\mathrm{Rel}(A, B) \cong \mathrm{Rel}(X, Y) \quad , \quad (A \rightharpoonup B) \cong (X \rightharpoonup Y) \quad ,$$

$$(A \Rightarrow B) \cong (X \Rightarrow Y) \quad , \quad \mathrm{Bij}(A, B) \cong \mathrm{Bij}(X, Y)$$

- $A \cong [1] \times A$, $(A \times B) \times C \cong A \times (B \times C)$, $A \times B \cong B \times A$
- $[0] \uplus A \cong A$, $(A \uplus B) \uplus C \cong A \uplus (B \uplus C)$, $A \uplus B \cong B \uplus A$
- $[0] \times A \cong [0]$, $(A \uplus B) \times C \cong (A \times C) \uplus (B \times C)$
- $(A \Rightarrow [1]) \cong [1]$, $(A \Rightarrow (B \times C)) \cong (A \Rightarrow B) \times (A \Rightarrow C)$
- $([0] \Rightarrow A) \cong [1]$, $((A \uplus B) \Rightarrow C) \cong (A \Rightarrow C) \times (B \Rightarrow C)$
- $([1] \Rightarrow A) \cong A$, $((A \times B) \Rightarrow C) \cong (A \Rightarrow (B \Rightarrow C))$
- $(A \rightharpoonup B) \cong (A \Rightarrow (B \uplus [1]))$
- $\mathcal{P}(A) \cong (A \Rightarrow [2])$

c.f. $c^{a.b} = (c^a)^b$

cf $c^{a+b} = c^a . c^b$

cf $(b.c)^a = b^a . c^a$

$$\begin{tikzcd} (\alpha * \beta \to \gamma) \arrow[rr, bend left, "\text{currying}"] & & (\alpha \to \beta \to \gamma) \arrow[ll, bend left, "\text{uncurrying}"] \end{tikzcd}$$

fun currying $(f : \alpha * \beta \to \gamma)\ (x : \alpha)\ (y : \beta)$

$\quad = f(x, y)$

fun uncurring $(g : \alpha \to \beta \to \gamma)\ ((x, y) : \alpha * \beta)$

$\quad = g\ x\ y$

Characteristic (or indicator) functions

$$\mathcal{P}(A) \cong (A \Rightarrow [2])$$

$[2] = \{0,1\}$

$$S \in \mathcal{P}(A) \longmapsto \chi_S : A \to \{0,1\}$$

$$\hat{f} \in \mathcal{P}(A) \longleftarrow\!\shortmid f : A \to \{0,1\}$$

$\forall a \in A.$

$$\chi_S(a) \stackrel{\text{def}}{=} \begin{cases} 0 & a \notin S \\ 1 & a \in S \end{cases}$$

$$\hat{f} \stackrel{\text{def}}{=} \{ a \in A \mid f(a) = 1 \} \in \mathcal{P}(A)$$

Exercise $\forall S \in \mathcal{P}(A).\quad \widehat{\chi_S} = S$

$\forall f \in (A \Rightarrow [2]).\quad \chi_{\hat{f}} = f$

$A = \{a_1, \ldots, a_n\}$ $\qquad$ $\#A = n$

$S \subseteq A$ $\qquad$ $S = \{a_1, a_3, \ldots a_{2i+1}, \ldots\}$

χ_S

$$\begin{array}{cccccc}
a_1 & a_2 & a_3 & a_4 & a_5 & \ldots \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
1 & 0 & 1 & 0 & 1 & \ldots
\end{array}$$

Finite cardinality

Definition 136 *A set A is said to be* finite *whenever $A \cong [n]$ for some $n \in \mathbb{N}$, in which case we write $\#A = n$.*

Theorem 137 *For all* $m, n \in \mathbb{N}$,

1. $\mathcal{P}\big([n]\big) \cong [2^n]$
2. $[m] \times [n] \cong [m \cdot n]$
3. $[m] \uplus [n] \cong [m + n]$
4. $\big([m] \rightharpoonup [n]\big) \cong \big[(n+1)^m\big]$
5. $\big([m] \Rightarrow [n]\big) \cong [n^m]$
6. $\mathrm{Bij}\big([n], [n]\big) \cong [n!]$

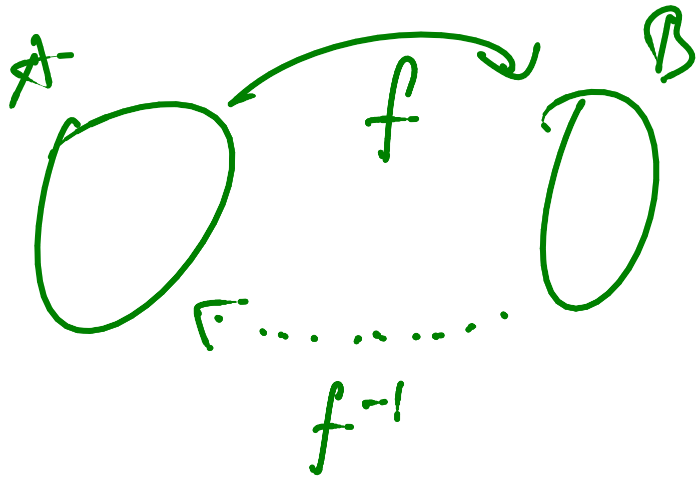
Infinity axiom

$\boxed{\text{There is an infinite set, containing } \emptyset \text{ and closed under successor.}}$

def iff def $\exists g: B \to A.\ g \circ f = id_A \wedge f \circ g = id_B$

Bijections

Proposition 138 *For a function* $f : A \to B$*, the following are equivalent.*



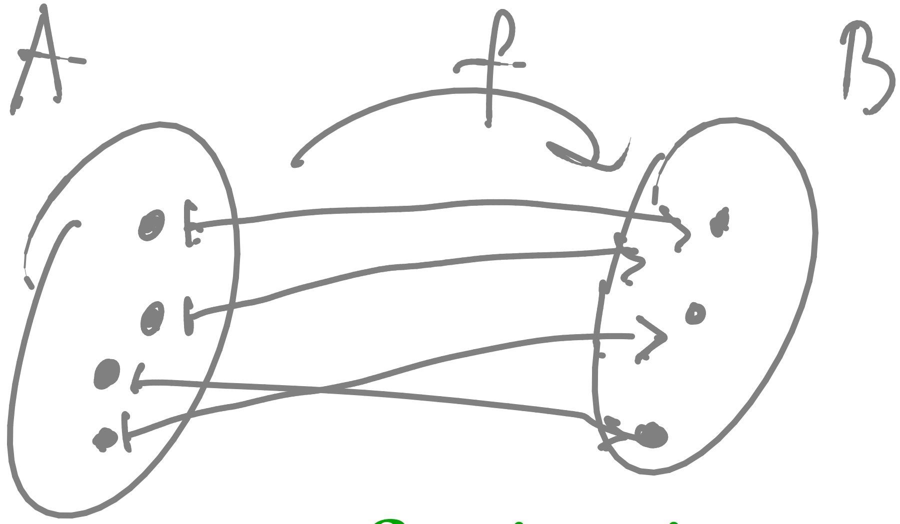
1. f *is bijective.* namely, $f^{-1}(a)$

2. $\forall\, b \in B.\ \exists!\, a \in A.\ f(a) = b$.

3. $\big(\forall\, b \in B.\ \exists\, a \in A.\ f(a) = b\big)$
$\wedge$
$\big(\forall\, a_1, a_2 \in A.\ f(a_1) = f(a_2) \implies a_1 = a_2\big)$

surjection

injection



Surjections

Definition 139 *A function* $f : A \to B$ *is said to be* surjective, *or a* surjection, *and indicated* $f : A \twoheadrightarrow B$ *whenever*

$$\forall\, b \in B.\, \exists\, a \in A.\, f(a) = b \quad .$$

$$\begin{tikzcd} A \arrow[r, two heads, "f"] \arrow[rr, bend right, "g\circ f"'] & B \arrow[r, two heads, "g"] & C \end{tikzcd}$$

Theorem 140 *The identity function is a surjection, and the composition of surjections yields a surjection.*

Let $c \in C$.
Find $a \in A$ s.t.
$(g\circ f)(a) = c$

The set of surjections from A to B is denoted

$$\mathrm{Sur}(A, B)$$

g surj $\Rightarrow \exists b \in B.\ g(b) = c$
Let $b_0 \in B.\ g(b_0) = c$

and we thus have

$$\mathrm{Bij}(A, B) \subseteq \mathrm{Sur}(A, B) \subseteq \mathrm{Fun}(A, B) \subseteq \mathrm{PFun}(A, B) \subseteq \mathrm{Rel}(A, B) .$$

f surj $\Rightarrow \exists a \in A.\ f(a) = b_0$
Let $a_0 \in A.\ f(a_0) = b_0$

Then $(g\circ f)(a_0) = g(f(a_0))$
$= g(b_0) = c$ $\boxtimes$

Enumerability

Definition 142

1. *A set $\mathbb{A}$ is said to be* enumerable *whenever there exists a surjection $\mathbb{N} \underset{e}{\twoheadrightarrow} \mathbb{A}$, referred to as an* enumeration*.*

2. *A* countable *set is one that is either empty or enumerable.*

idea $\{e(0), e(1), e(2), \ldots, e(n), \ldots\} = A$

$\{e(n) \mid n \in \mathbb{N}\} = A$

Examples:

1. A bijective enumeration of $\mathbb{Z}$.

$\cdots$	-3	-2	-1	0	1	2	3	$\cdots$
	$\ldots$	3	1	0	2	4	$\ldots$	$\ldots$

$\uparrow e$

$$n = 2^{x} \cdot \underbrace{\boxed{3^{\square}\, 5^{\square} \cdots\, p_k^{\square}}}_{\text{odd}}$$

2. A bijective enumeration of $\mathbb{N} \times \mathbb{N}$.

Fact: for all $n \in \mathbb{N}^+$ $\exists!\, x, y \in \mathbb{N}$. $n = 2^{x} \cdot (2y+1)$

$f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}^+$

$f(x,y) = 2^{x}(2y+1)$

is bijective

	0	1	2	3	4	5	...
0							
1							
2							
3							
4							
⋮							

$g : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$

$g(x,y) = 2^{x}(2y+1) - 1$ is bijective.

Proposition 143 *Every non-empty subset of an enumerable set is enumerable.*

$\exists s \in S$

PROOF: Let A be enumerable; so let $e : \mathbb{N} \twoheadrightarrow A$ be an enumeration.

Let $\emptyset \neq S \subseteq A$. Claim: S is enumerable.

RTP: $\exists f : \mathbb{N} \twoheadrightarrow S$

Idea:

Def $A \overset{h}{\twoheadrightarrow} S$

$\mathbb{N} \overset{e}{\twoheadrightarrow} A \overset{h}{\twoheadrightarrow} S$

$f = h \circ e$

$\forall a \in A.$

$$h(a) \stackrel{\text{def}}{=} \begin{cases} a, & a \in S \\ s, & a \notin S \end{cases}$$

$\forall n \in \mathbb{N}$

$$f(n) \stackrel{\text{def}}{=} \begin{cases} e(n), & e(n) \in S \\ s, & \text{otw} \end{cases}$$

Countability

Proposition 144

1. *$\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ are countable sets.*
2. *The product and disjoint union of countable sets is countable.*
3. *Every finite set is countable.*
4. *Every subset of a countable set is countable.*