Bijections

= reversible functions

$A \xrightarrow{f} B$ is a bijection

Iff $\exists.\ g : B \to A$ a function

s.t.

$g \circ f = \mathrm{id}_A$ $\left(\forall a \in A.\ g(f(a)) = a \right)$

and

$f \circ g = \mathrm{id}_B$ $\left(\forall b \in B.\ f(g(b)) = b \right)$

NB: If such a g exists then it is unique depending on f and we typically denote it f^{-1}

Given $f: A \to B$ suppose $g, h: B \to A$ s.t. and $f \circ g = f \circ h = id_B$
$g \circ f = h \circ f = id_A$

<u>RTP</u>: $g = h$

<u>Idea</u>:

$$
\begin{array}{ccccc}
f \circ g = f \circ h & \Rightarrow & h \circ f \circ g & = & h \circ f \circ h \\
& & \| & & \| \\
& & id_A \circ g & & h \circ id_B \\
& & \| & & \| \\
& & g & & h
\end{array}
$$

$\boxtimes$

Bijections

Definition 127 *A function $f : A \longrightarrow B$ is said to be* bijective*, or a* bijection*, whenever there exists a (necessarily unique) function $g : B \longrightarrow A$ (referred to as the* inverse *of f) such that*

1. *g is a* retraction *(or* left inverse*) for f:*

   $$g \circ f = \mathrm{id}_A \quad ,$$

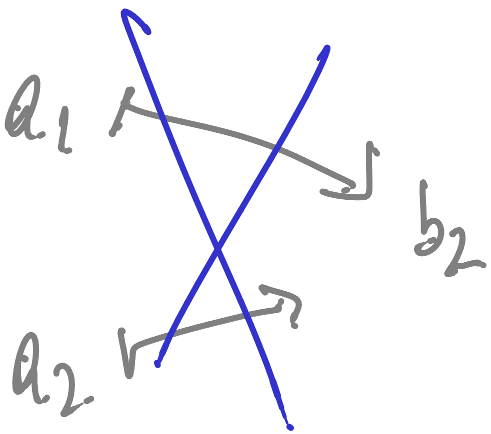
2. *g is a* section *(or* right inverse*) for f:*

   $$f \circ g = \mathrm{id}_B \quad .$$

$A = \{a_1, a_2, a_3\}$ $\qquad$ $B = \{b_1, b_2\}$

$A \xrightarrow{f} B$ bijection $\qquad$ $B \xrightarrow{g} A$

$$
\begin{aligned}
a_1 &\longmapsto b_2 \\
a_2 &\longmapsto b_1 \\
a_3 &\longmapsto ?
\end{aligned}
\qquad\qquad
\begin{aligned}
b_1 &\longmapsto a_2 \\
b_2 &\longmapsto a_1
\end{aligned}
$$

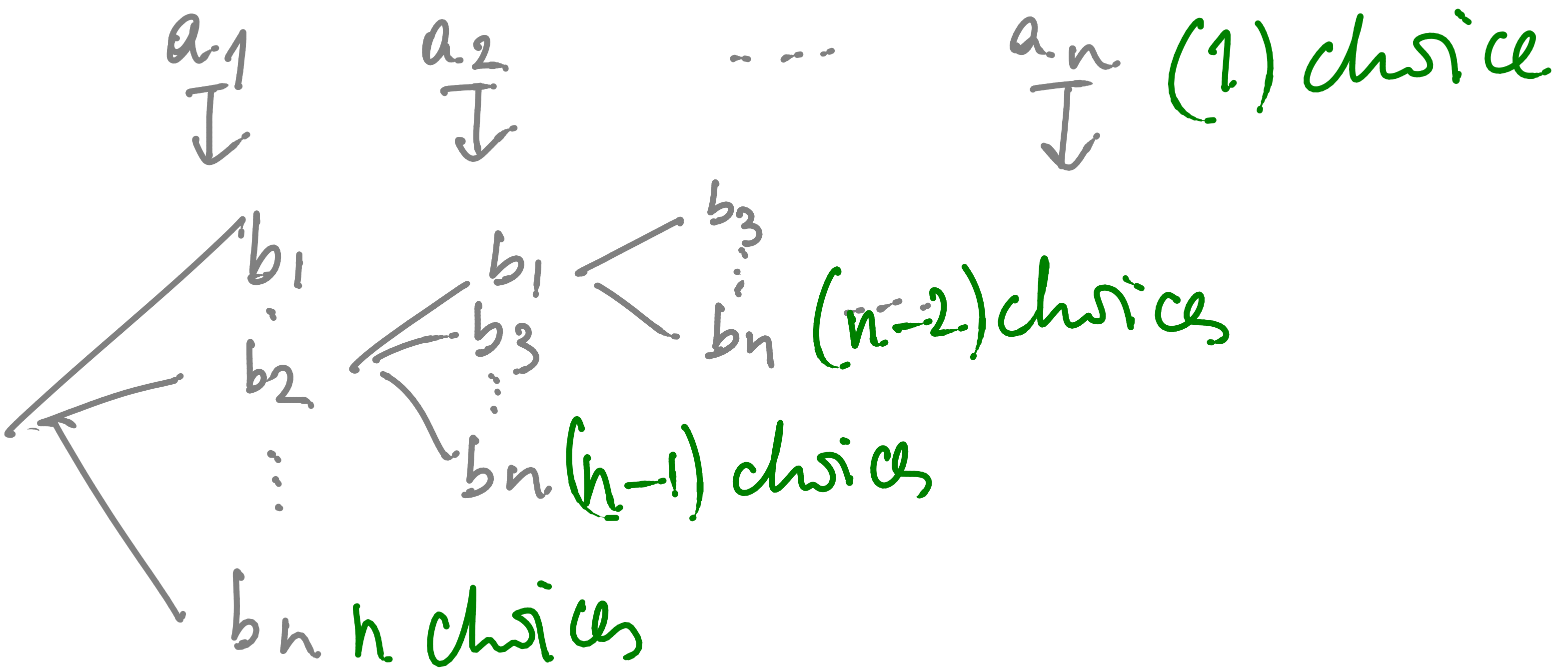


Proposition 129 *For all finite sets* A *and* B,

$$\#\mathrm{Bij}(A, B) = \begin{cases} 0 & \text{, if } \#A \neq \#B \\ n! & \text{, if } \#A = \#B = n \end{cases}$$ ✓

PROOF IDEA:

$A = \{a_1, \ldots, a_n\}$ $\quad B = \{b_1, \ldots, b_n\}$



Theorem 130 *The identity function is a bijection, and the composition of bijections yields a bijection.*

$A \underset{\text{bij}}{\xrightarrow{f}} B \underset{\text{bij}}{\xrightarrow{g}} C \implies g \circ f : A \underset{\text{bij.}}{\longrightarrow} C$

Ex: Check that

$(g \circ f)^{-1} = f^{-1} \circ g^{-1}$

Definition 131 *Two sets* A *and* B *are said to be* isomorphic *(and to have the* same cardinatity*) whenever there is a bijection between them; in which case we write*

$$A \cong B \quad \textit{or} \quad \#A = \#B \quad .$$

Counterexample

$\mathbb{N} \not\cong \mathbb{R}$

Examples:

1. $\{0, 1\} \cong \{\mathbf{false}, \mathbf{true}\}$.

2. $\mathbb{N} \cong \mathbb{N}^+$, $\mathbb{N} \cong \mathbb{Z}$, $\mathbb{N} \cong \mathbb{N} \times \mathbb{N}$, $\mathbb{N} \cong \mathbb{Q}$.

- $\mathbb{N} \cong \mathbb{N}^+ = \{ n \in \mathbb{N} \mid n > 0 \}$

$$\begin{tikzcd} \mathbb{N} \arrow[r, bend left, "f"] \arrow[r, phantom, "\cong"] & \mathbb{N}^+ \arrow[l, bend left, "g"] \end{tikzcd}$$

$f(n) \overset{\text{def}}{=} n+1 \qquad n \in \mathbb{N}$

$g(m) \overset{\text{def}}{=} m-1 \qquad m \in \mathbb{N}^+$

- $\begin{tikzcd} \mathbb{N} \arrow[r, bend left, "f"] \arrow[r, phantom, "\cong"] & \mathbb{Z} \arrow[l, bend left, "g"] \end{tikzcd}$

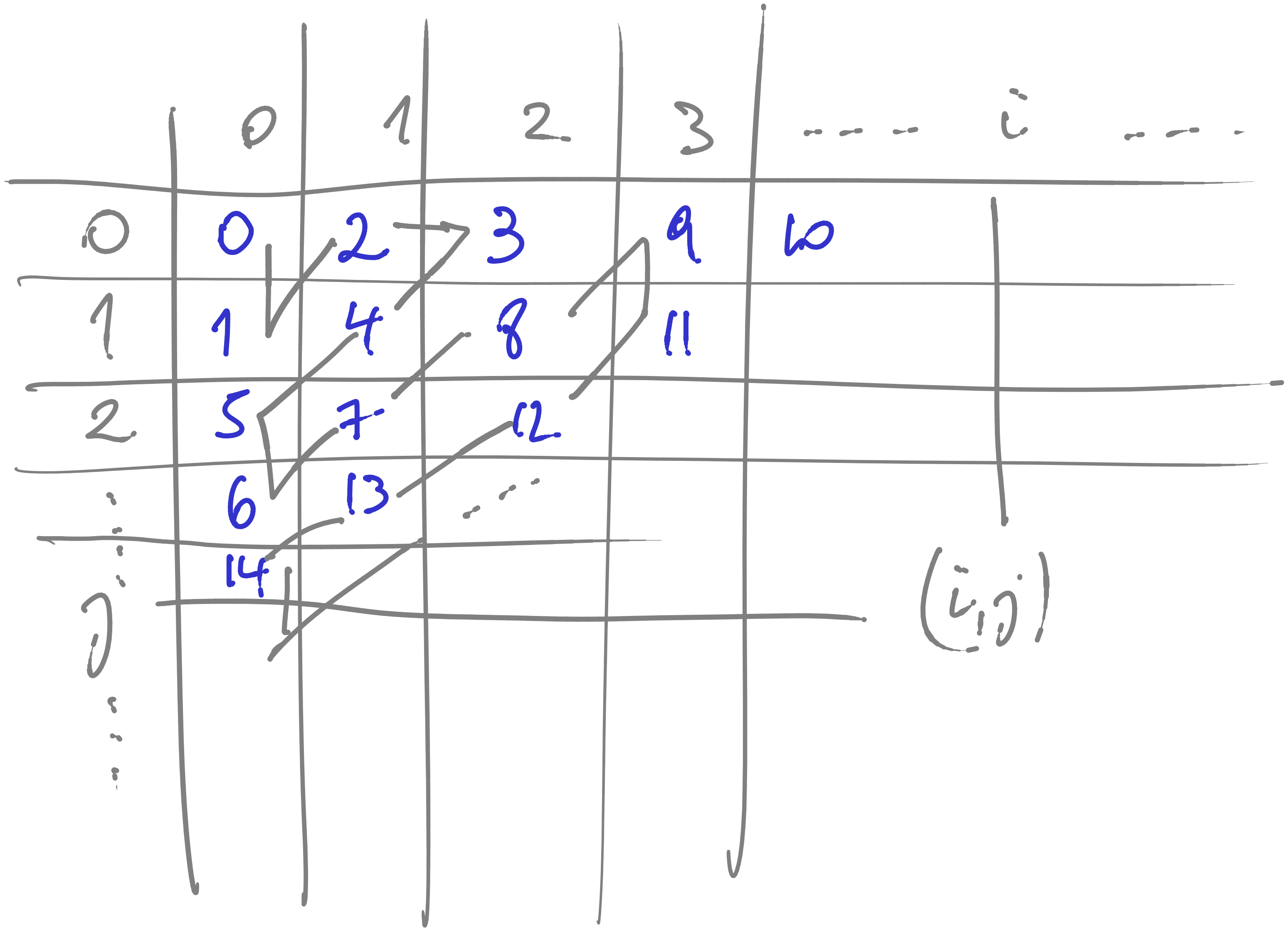
$0 \quad 1 \quad 2 \quad 3 \quad \cdots$

$\cdots$

$\cdots \quad -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \quad \cdots$

$$f(n) = \begin{cases} n/2 & n \text{ even} \\ -(n+1/2) & n \text{ odd} \end{cases}$$

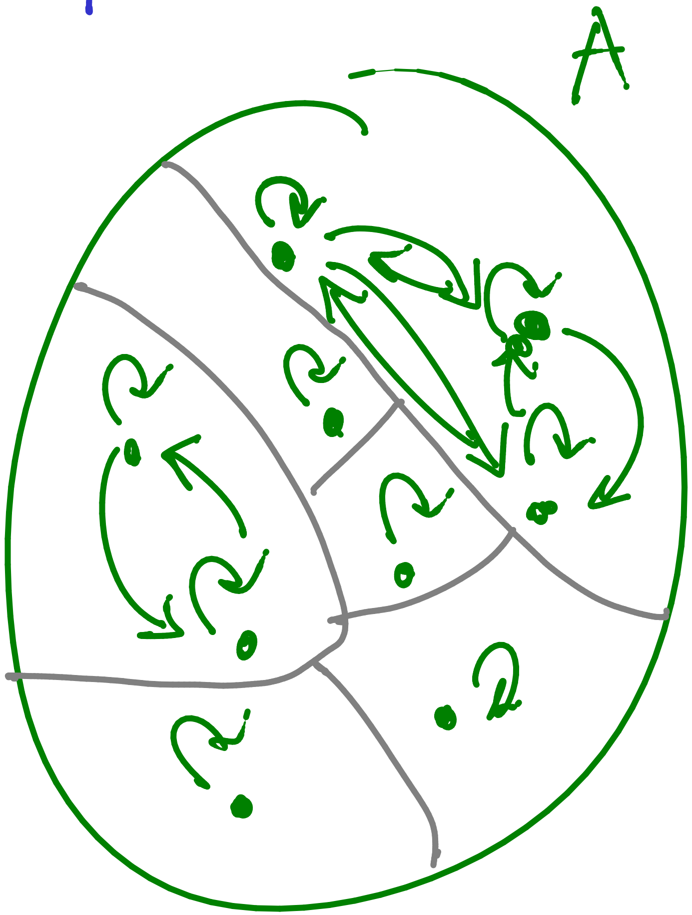
- $\mathbb{N} \cong \mathbb{N}\times\mathbb{N}$



Equivalence relations and set partitions

- Equivalence relations.

$$Eq(A) = \{ R \subseteq A \times A \mid R \text{ is reflexive}, R \text{ is transitive}, R \text{ is symmetric} \}$$



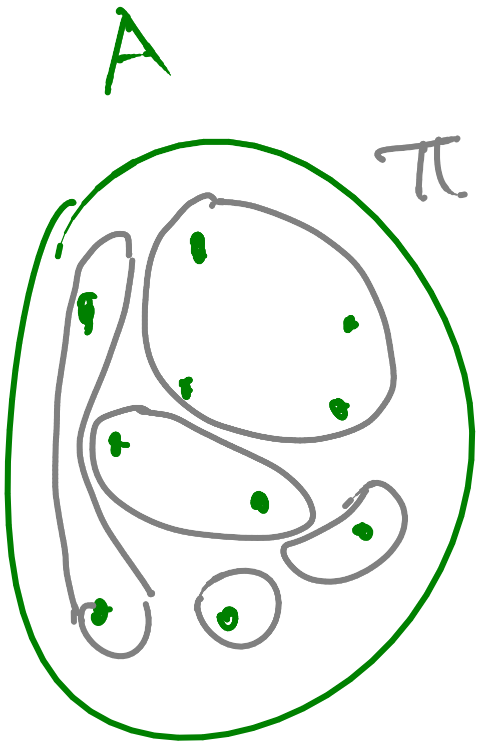
$\forall (a, a') \in A \times A.\ a R a' \Rightarrow a' R a$

- Set partitions.

$\mathrm{Part}(A)$ the set of all partitions of A.

$$= \{\pi \subseteq \mathcal{P}(A) \mid \pi \text{ is a partition}\}$$

?



$\emptyset \notin \pi$

$\bigcup \pi = A$

$\forall b_1, b_2 \in \pi.\ b_1 \neq b_2 \Longrightarrow b_1 \cap b_2 = \emptyset$

$$\begin{tikzcd} Eq(A) \arrow[r, bend left, "f"] \arrow[r, phantom, "\cong"] & Part(A) \arrow[l, bend left, "g"] \end{tikzcd}$$

- Define f:

Given an arbitrary equiv. rel R on A we define $\pi(R) \subseteq P(A)$ and show it is a partition. Exercise.

$$\pi(R) = \{ [a]_R \subseteq A \mid a \in A \}$$

$/\!/_{def}$ the equivalence class of $a \in A$.

$$\{x \in A \mid x R a\}$$

NB $a \in [a]_R$

- Define g:

Given an arbitrary partition π of A
define $eq(\pi) \subseteq A \times A$ s.t. it is an eq. rel.

Exercise.

$a, a' \in A$

$$(a, a') \in eq(\pi) \overset{\text{def}}{\iff} \exists b \in \pi.\ a \in b \wedge a' \in b$$

Exercises: $\forall R \in Eq(A)$ $\quad g(f(R)) = R$

$\forall \pi \in Part(A)$ $\quad f(g(\pi)) = \pi$

Theorem 134 *For every set A,*

$$\mathrm{EqRel}(A) \cong \mathrm{Part}(A) \quad .$$

PROOF: