$\bigcap \mathcal{F}_R \subseteq R^{\circ *}$

Theorem 118 *For* $R \subseteq A \times A$*, let*

$$\mathcal{F}_R = \{ Q \subseteq A \times A \mid R \subseteq Q \wedge Q \textit{ is a preorder} \} \quad .$$

Then, (i) $R^{\circ *} \in \mathcal{F}_R$ *and* (ii) $R^{\circ *} \subseteq \bigcap \mathcal{F}_R$. *Hence,* $R^{\circ *} = \bigcap \mathcal{F}_R$.

PROOF:

(i) $R^{\circ *} \subseteq A \times A$

$R \subseteq R^{\circ *} = \bigcup_{n \in \mathbb{N}} R^{\circ n}$ $\qquad R^{\circ 1} = R$

$R^{\circ *}$ is a preorder

- reflexivity: $id_A \subseteq R^{\circ *}$ $\qquad R^{\circ \emptyset} = id_A$

- transitivity: $R^{\circ *} \circ R^{\circ *} \subseteq R^{\circ *}$

$\forall x, y, z.\ x R^{\circ *} y \wedge y R^{\circ *} z \Rightarrow x R^{\circ *} z.$

$R^{\circ *} \subseteq \bigcap \mathcal{F}_R$

$\iff$

$R^{\circ *} \subseteq Q \quad \forall Q \subseteq A \times A .\ R \subseteq Q \wedge Q \text{ a preorder.}$

$\shortparallel$

$\bigcup_{n \in \mathbb{N}} R^{\circ n}$

$\iff \left(\forall Q \subseteq A \times A .\ R \subseteq Q \wedge Q \text{ preorder} \Rightarrow \forall n \in \mathbb{N} .\ R^{\circ n} \subseteq Q \right)$

$X \subseteq \bigcap \{A_i \mid i \in I\}$

iff

$\forall i \in I .\ X \subseteq A_i$

$\bigcup \{A_i \mid i \in I\} \subseteq X$

iff

$\forall i \in I .\ A_i \subseteq X$

Let $Q \subseteq A \times A$ s.t. $R \subseteq Q \wedge Q$ is a preorder

We show $R^{\circ n} \subseteq Q$ for all $n \in \mathbb{N}$ by induction.

Base case: $R^{\circ 0} \subseteq Q$ ✓

$$R^{\circ 0} = id_A$$

Ind. Step: Assume $R^{\circ n} \subseteq Q$ for $n \in \mathbb{N}$. (IH)

RTP: $R^{\circ (n+1)} \subseteq Q$ ✓

$$R^{\circ(n+1)} = R \circ R^{\circ n}$$

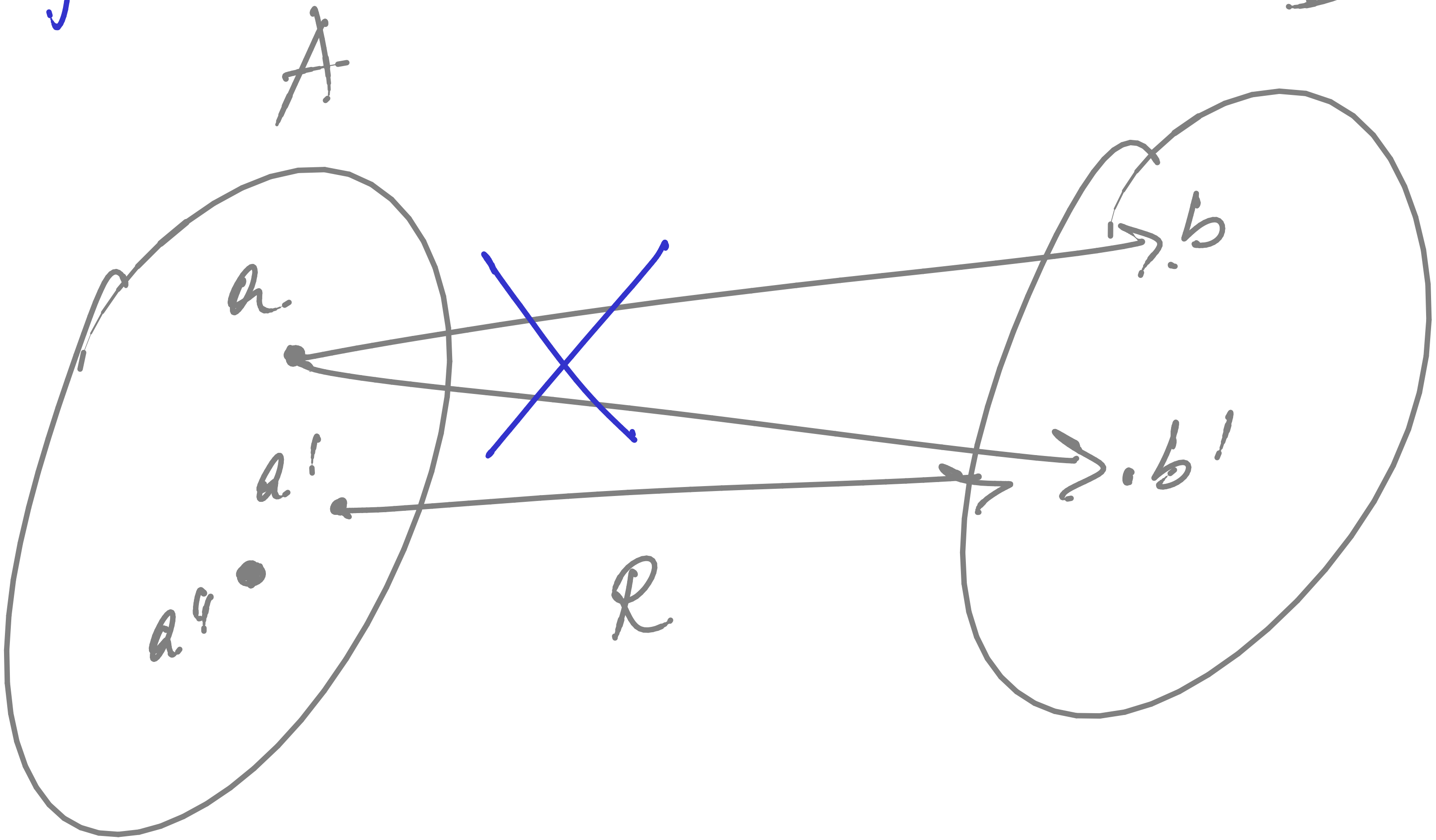
Let $x \, (R \circ R^{\circ n}) \, y \iff \exists z.\ x R z \wedge z R^{\circ n} y$

$\Rightarrow \exists z.\ x Q z \wedge z Q y \Rightarrow x Q y$ ☒

non-deterministic input-output behaviours.

$R : A \nrightarrow B$

functional



Partial functions

Definition 119 *A relation* $R : A \rightharpoonup B$ *is said to be* functional, *and called a* partial function, *whenever it is such that*

$$\forall a \in A. \forall b_1, b_2 \in B. \; a R b_1 \wedge a R b_2 \implies b_1 = b_2 \quad .$$

If $f : A \rightharpoonup B$ is a partial function

Then $\forall a \in A.$ if $a f b \in B$ Then such a b is unique.

NOTATION we typically denote it by $f(a)$

NOTATION

$f(a)\downarrow$ iff $\exists b.\, a f b$

$f(a)\uparrow$ iff $\neg(f(a)\downarrow)$

EXAMPLE: id_A is a partial function.

NOTATION

$f : A \to B$ partial functions.

$g : B \to C$

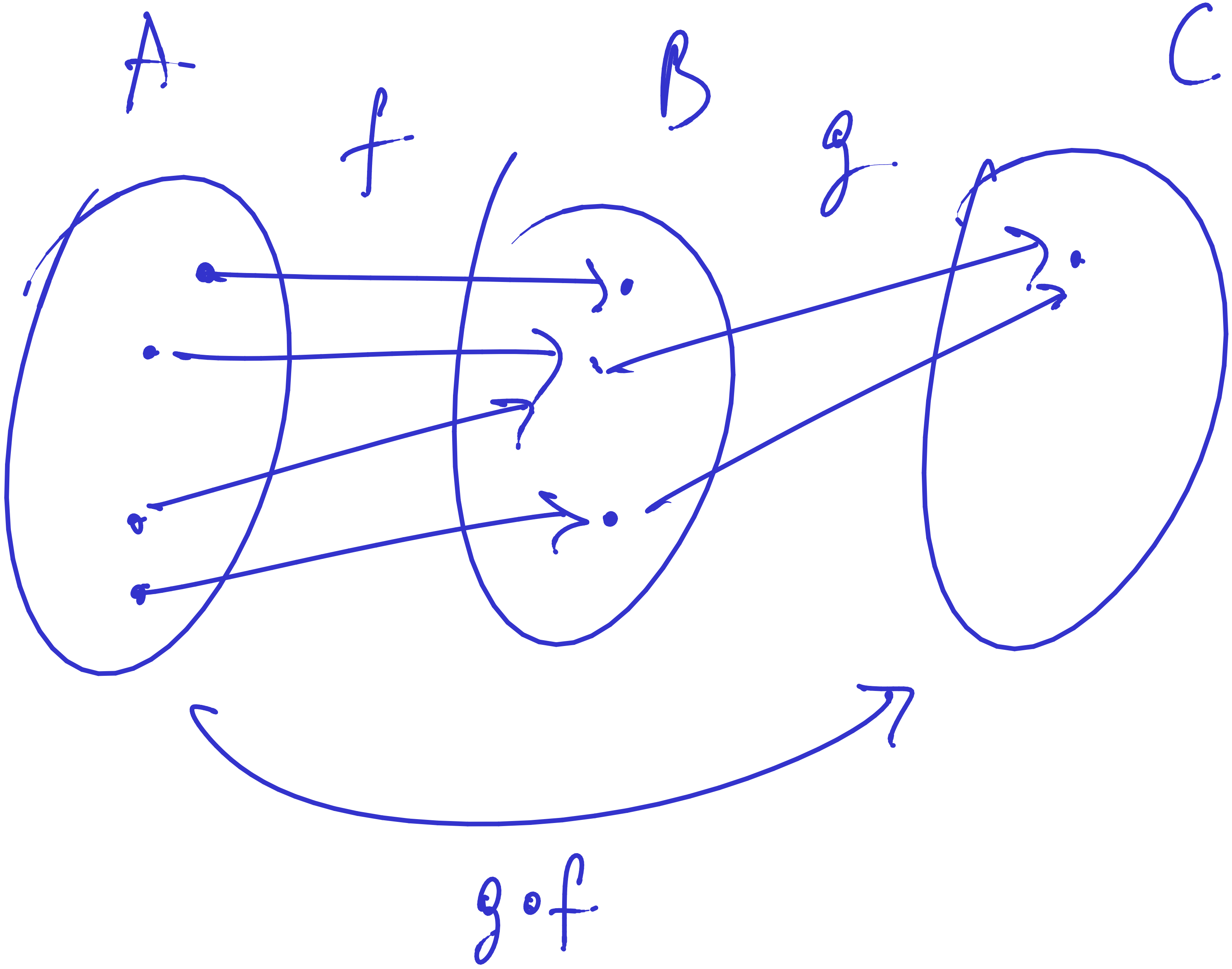
So

$g \circ f : A \rightarrowtail C$ is it functional ?

YES !

$a \in A$

$$(g \circ f)(a) = \begin{cases} \uparrow & \text{if } f(a)\uparrow \\ \uparrow & \text{if } f(a)\downarrow \text{ but } g(f(a))\uparrow \\ g(f(a)) & \text{if } f(a)\downarrow \text{ and } g(f(a))\downarrow \end{cases}$$



A
f
B
g
C
g∘f

Theorem 121 *The identity relation is a partial function, and the composition of partial functions yields a partial function.*

NB

$$f = g : A \rightharpoonup B$$

iff

$$\forall a \in A. \left(f(a){\downarrow} \iff g(a){\downarrow} \right) \wedge f(a) = g(a)$$

Convention: Typically one defines partial functions $f : A \rightharpoonup B$ by selecting $D \subseteq A$ (domain of definition) and giving a rule/mapping that associates $a \in D$ to $f(a) \in B$

$$\left(a \overset{f}{\longmapsto} f(a) \right).$$

Recall $\mathbb{N} \times \mathbb{N} \to \mathbb{N} \times \mathbb{N} : (n, m) \mapsto (\underline{quo}(n, m), rem(n, m))$

Example: The following defines a partial function $\mathbb{Z} \times \mathbb{Z} \rightharpoonup \mathbb{Z} \times \mathbb{N}$:

- for $n \geq 0$ and $m > 0$,
  $$(n, m) \mapsto \big(\, \mathrm{quo}(n, m)\, ,\ \mathrm{rem}(n, m)\, \big)$$
- for $n \geq 0$ and $m < 0$,
  $$(n, m) \mapsto \big(\, -\mathrm{quo}(n, -m)\, ,\ \mathrm{rem}(n, -m)\, \big)$$
- for $n < 0$ and $m > 0$,
  $$(n, m) \mapsto \big(\, -\mathrm{quo}(-n, m) - 1\, ,\ \mathrm{rem}(m - \mathrm{rem}(-n, m), m)\, \big)$$
- for $n < 0$ and $m < 0$,
  $$(n, m) \mapsto \big(\, \mathrm{quo}(-n, -m) + 1\, ,\ \mathrm{rem}(-m - \mathrm{rem}(-n, -m), -m)\, \big)$$

Its domain of definition is $\{\, (n, m) \in \mathbb{Z} \times \mathbb{Z} \mid m \neq 0 \,\}$.

$(A \rightharpoonup B) =$ the set of all partial functions from A to B

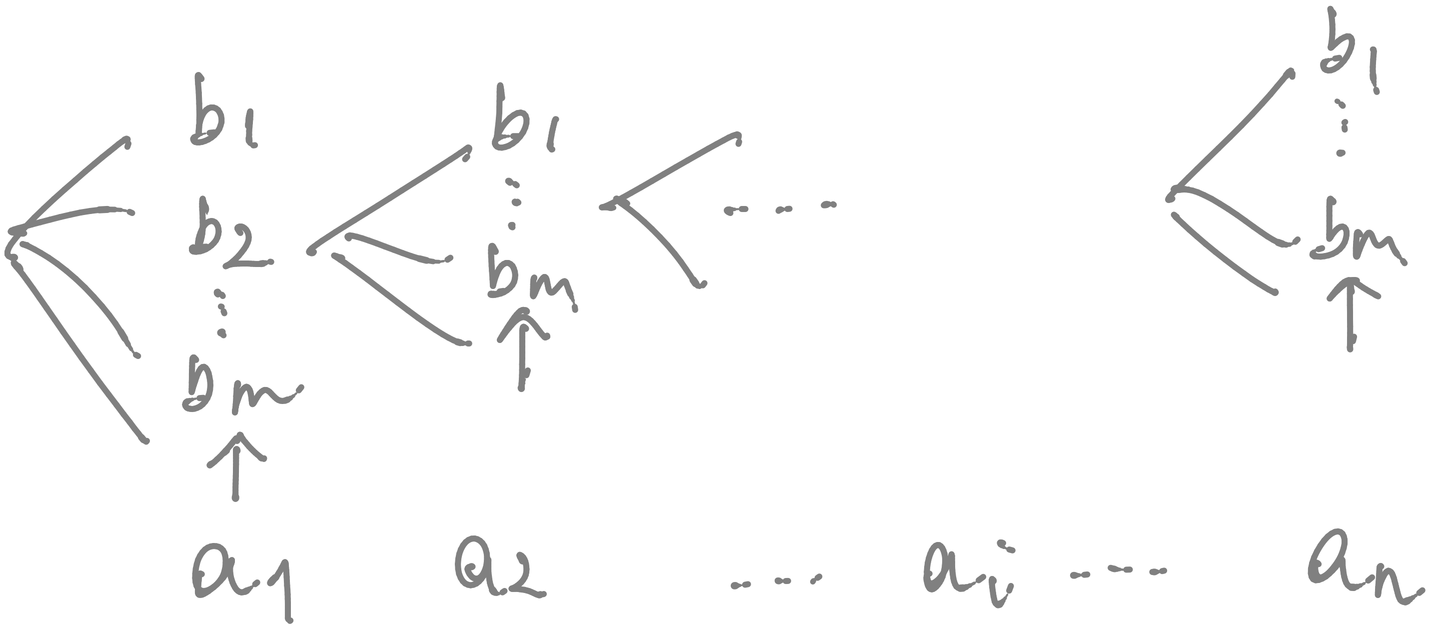
Proposition 122 *For all finite sets* A *and* B,

$$\#(A \rightharpoonup B) = (\#B+1)^{\#A} \quad . \quad = (m+1)^n$$

PROOF IDEA:

$A = \{a_1, \ldots, a_n\}$ $\quad \#A = n$

$B = \{b_1 \ldots b_m\}$ $\quad \#B = m$



Functions (or maps)

Definition 123 *A partial function is said to be* total*, and referred to as a* (total) function *or* map*, whenever its domain of definition coincides with its source.*

$f : A \to B$ is total

iff

$\underline{\mathrm{dom}}(f) = \{ a \in A \mid f(a)\downarrow \} = A$

NOTATION: For these we write $f : A \to B$.

$(A \Rightarrow B)$ = set of all functions from A to B.

Theorem 124 *For all* $f \in \mathrm{Rel}(A, B)$,

$$f \in (A \Rightarrow B) \iff \forall\, a \in A.\, \exists!\, b \in B.\ \underbrace{a\, f\, b}_{}\quad .$$

the unique such b is denoted $f(a)$.

Proposition 125 *For all finite sets* A *and* B,

$$\#(A \Rightarrow B) \;=\; \#B^{\#A} \;=\; m^n$$

PROOF IDEA:

$A = \{ a_1 \text{ — } a_n \}$

$B = \{ b_1 \text{ — } b_m \}$

$a_1 \mapsto \quad b_1 \text{ — } b_m$

$a_2 \mapsto \quad b_1 \cdots b_m$

$\vdots \qquad\qquad \vdots$

$a_n \mapsto \quad b_1 \cdots b_m$

NB

$\#(A \Rightarrow B)$

$= \#(A \Rightarrow (B \uplus [1]))$

$[n] = \{0, \ldots, n-1\}$

$[1] = \{0\}$

Theorem 126 *The identity partial function is a function, and the composition of functions yields a function.*

NB

1. $f = g : A \longrightarrow B$ iff $\forall\, a \in A.\, f(a) = g(a)$.

2. For all sets A, the identity function $\mathrm{id}_A : A \longrightarrow A$ is given by the rule

   $$\mathrm{id}_A(a) = a$$

   and, for all functions $f : A \longrightarrow B$ and $g : B \longrightarrow C$, the composition function $g \circ f : A \longrightarrow C$ is given by the rule

   $$(g \circ f)(a) = g\big(f(a)\big) \quad .$$