Definition 103

1. For positive integers $m$ and $n$, an $(m \times n)$-matrix $M$ over a semiring $(S, 0, \oplus, 1, \odot)$ is given by entries $M_{i,j} \in S$ for all $0 \leq i < m$ and $0 \leq j < n$.

\[
M = (m \times n)\text{-matrix, } N = (n \times l)\text{-matrix}
\]

\[
(M_{ij})_{0 \leq i < m, 0 \leq j < n}, \quad (N_{jk})_{0 \leq j < n, 0 \leq k < l}
\]

\[
(N \cdot M) = (m \times l)\text{-matrix}
\]

\[
(N \cdot M)_{i,k} = \bigoplus_{0 \leq j \leq n-1} M_{ij} \odot N_{jk}
\]

Theorem 104 Matrix multiplication is associative and has the identity matrix as neutral element.
Relations from $[m]$ to $[n]$ and $(m \times n)$-matrices over Booleans provide two alternative views of the same structure.

This carries over to identities and to composition/multiplication.

Given $R \subseteq [m] \times [n]$

\[
\text{def. } \text{mat}(R) \ (m \times n) \text{- matrix } \\
(\text{mat}(R))_{i,j} = \begin{cases} 
\text{true} & \text{if } (i,j) \in R \\
\text{false} & \text{otherwise}
\end{cases}
\]
Given \( M(m \times n) \)-matrices

\[ \text{def. } \quad \text{rel}(M) \subseteq [m] \times [n] \]

\[ \forall i \in [m], j \in [n]. \]

\[ (i, j) \in \text{rel}(M) \quad \text{if and only if} \quad M_{i,j} = \text{true} \]

Claim \( \subseteq [m] \times [n] \)

\[ R \rightarrow \text{mat}(R) \rightarrow \text{rel}(\text{mat}(R)) = R \]

\( (m \times n) \)-mat.

\[ M \rightarrow \text{rel}(M) \rightarrow \text{mat}(\text{rel}(M)) = M \]

\( \text{mat}(\text{rel}(M))_{i,j} = \text{true} \iff (i,j) \in \text{rel}(M) \iff M_{i,j} = \text{true} \)
\[ R \subseteq [m] \times [n] \quad S \subseteq [n] \times [e] \]

\[ S \circ R \subseteq [m] \times [e] \]

\[ \text{mat} (R) \ (m \times n) \quad \text{mat} (S) \ (n \times e) \]

\[ \text{mat} (S) \ast \text{mat} (R) \]

\[ \Rightarrow \text{claim} \]

\[ \text{mat} (S \circ R) \]
Directed graphs

Definition 108 A directed graph \((A, R)\) consists of a set \(A\) and a relation \(R\) on \(A\) (i.e. a relation from \(A\) to \(A\)).
Corollary 110  For every set $A$, the structure

$$( \text{Rel}(A), \text{id}_A, \circ )$$

is a monoid.

Definition 111  For $R \in \text{Rel}(A)$ and $n \in \mathbb{N}$, we let

$R^n = \underbrace{R \circ \cdots \circ R}_{\text{n times}} \in \text{Rel}(A)$

be defined as $\text{id}_A$ for $n = 0$, and as $R \circ R^m$ for $n = m + 1$. 
Proposition 113  Let \((A, R)\) be a directed graph. For all \(n \in \mathbb{N}\) and \(s, t \in A\), \(s R^n t\) iff there exists a path of length \(n\) in \(R\) with source \(s\) and target \(t\).

**Proof:** By induction on \(n \in \mathbb{N}\).

- Base case: \(s R^0 t \iff \exists\ \text{a path of length 0 from } s \text{ to } t\).
- Ind. step: Assume \(s R^m t \iff \exists\ \text{a path of length } m \text{ from } s \text{ to } t\).

RTP: \(\forall s, t,\ s R^{(n+1)} t \iff \exists\ \text{a path of length } n+1 \text{ from } s \text{ to } t\).
\[ s \mathrel{R^{(n+1)}} t \iff s(R^*R^n) t \]

\[ \iff \exists u. sRu \land uR^nt \]

by IH

\[ \iff \exists u. sRu \land \exists \text{ path of length } n \text{ from } u \text{ to } t \]

\[ \iff \exists \text{ path of length } (n+1) \text{ from } s \text{ to } t. \]
**Definition 114** For $R \in \text{Rel}(A)$, let

$$R^{o*} = \bigcup \{ R^n \in \text{Rel}(A) \mid n \in \mathbb{N} \} = \bigcup_{n \in \mathbb{N}} R^n.$$

**Corollary 115** Let $(A, R)$ be a directed graph. For all $s, t \in A$, $s \ R^{o*} \ t$ iff there exists a path with source $s$ and target $t$ in $R$.

How do we calculate $R^o_n$? (n times)
The \((n \times n)\)-matrix \(M = \text{mat}(R)\) of a finite directed graph \([n], R\) for \(n\) a positive integer is called its \textit{adjacency matrix}.

The adjacency matrix \(M^* = \text{mat}(R^{\circ\ast})\) can be computed by matrix multiplication and addition as \(M_n\) where

\[
\begin{cases}
M_0 = I_n \\
M_{k+1} = I_n + (M \cdot M_k)
\end{cases}
\]

This gives an algorithm for establishing or refuting the existence of paths in finite directed graphs.
Definition 116  A preorder \((\mathcal{P}, \sqsubseteq)\) consists of a set \(\mathcal{P}\) and a relation \(\sqsubseteq\) on \(\mathcal{P}\) (i.e. \(\sqsubseteq \in \mathcal{P}(\mathcal{P} \times \mathcal{P})\)) satisfying the following two axioms.

- **Reflexivity.**

\[
\forall x \in \mathcal{P}. \ x \sqsubseteq x
\]

- **Transitivity.**

\[
\forall x, y, z \in \mathcal{P}. \ (x \sqsubseteq y \land y \sqsubseteq z) \implies x \sqsubseteq z
\]
Examples:

- \((\mathbb{R}, \leq)\) and \((\mathbb{R}, \geq)\).

- \((\mathcal{P}(A), \subseteq)\) and \((\mathcal{P}(A), \supseteq)\).

- \((\mathbb{Z}, |)\). 
  
  \[ n \mid n \]
  
  \[ n \mid m \land m \mid l \Rightarrow n \mid l. \]
Theorem 118 For $R \subseteq A \times A$, let $\mathcal{F}_R = \{ Q \subseteq A \times A \mid R \subseteq Q \wedge Q$ is a preorder $\}$. Then, (i) $R^\circ \in \mathcal{F}_R$ and (ii) $R^\circ \subseteq \bigcap \mathcal{F}_R$. Hence, $R^\circ = \bigcap \mathcal{F}_R$.

PROOF: