Relations and matrices

Definition 103

1. *For positive integers* m *and* n*, an* $(m \times n)$*-*<u>*matrix*</u> M *over a semiring* $(S, 0, \oplus, 1, \odot)$ *is given by entries* $M_{i,j} \in S$ *for all* $0 \leq i < m$ *and* $0 \leq j < n$*.*

M $(m \times n)$-matrix, N $(n \times l)$-matrix

$M = (M_{i,j})_{\substack{0 \leq i < m \\ 0 \leq j < n}}$ $\qquad N = (N_{j,k})_{\substack{0 \leq j < n \\ 0 \leq k < l}}$

$(N * M)$ $(m \times l)$-matrix

$$(N * M)_{i,k} = \bigoplus_{j=0}^{n-1} M_{i,j} \odot N_{j,k} \qquad \begin{array}{l} 0 \leq i < m \\ 0 \leq k < l \end{array}$$

Theorem 104 *Matrix multiplication is associative and has the identity matrix as neutral element.*

$$R \subseteq \{0, 1, \ldots, m-1\} \times \{0, 1, \ldots, n-1\}$$

Relations from $[m]$ to $[n]$ and $(m \times n)$-matrices over Booleans provide two alternative views of the same structure.

This carries over to identities and to composition/multiplication .

$$(N * M)_{i,k} = \bigvee_{j=0}^{n-1} (M_{i,j} \wedge N_{j,k}) \quad \rightsquigarrow$$

$$\mathbb{B} = \{\text{true}, \text{false}\}$$
$$\oplus = \vee$$
$$\odot = \wedge$$

Given $R \subseteq [m] \times [n]$

def. $\underline{mat}(R)$ $(m \times n)$-matrix

$$(\underline{mat}(R))_{ij} = \begin{cases} \text{true} & \text{if } (i,j) \, R \\ \text{false} & \text{otw.} \end{cases}$$

Given M $(m \times n)$-matrix

def. $\underline{\mathrm{rel}}(M) \subseteq [m] \times [n]$

$\forall i \in [m], j \in [n].$

$$(i,j) \in \underline{\mathrm{rel}}(M) \text{ iff}_{\mathrm{def}} \ M_{i,j} = \mathrm{true}$$

__Claim__

$$\overset{\subseteq [m]\times[n]}{R} \rightsquigarrow \underline{\mathrm{mat}}(R) \rightsquigarrow \underline{\mathrm{rel}}(\underline{\mathrm{mat}}(R)) \overset{\subseteq [m]\times[n]}{=} R$$

$$M^{(m\times n)\text{-mat}} \rightsquigarrow \underline{\mathrm{rel}}(M) \rightsquigarrow \underline{\mathrm{mat}}(\underline{\mathrm{rel}}(M))^{(m\times n)\text{-mat}} = M$$

$$\underline{\mathrm{mat}}(\underline{\mathrm{rel}}(M))_{i,j} = \mathrm{true} \Leftrightarrow (i,j) \in \underline{\mathrm{rel}}(M) \Leftrightarrow M_{ij} = \mathrm{true}$$

$R \subseteq [m] \times [n]$ $\qquad$ $S \subseteq [n] \times [\ell]$

$S \circ R \subseteq [m] \times [\ell]$

$\underline{\text{mat}}(R)$ $(m \times n)$-mat. $\qquad$ $\underline{\text{mat}}(S)$ $(n \times \ell)$-mat

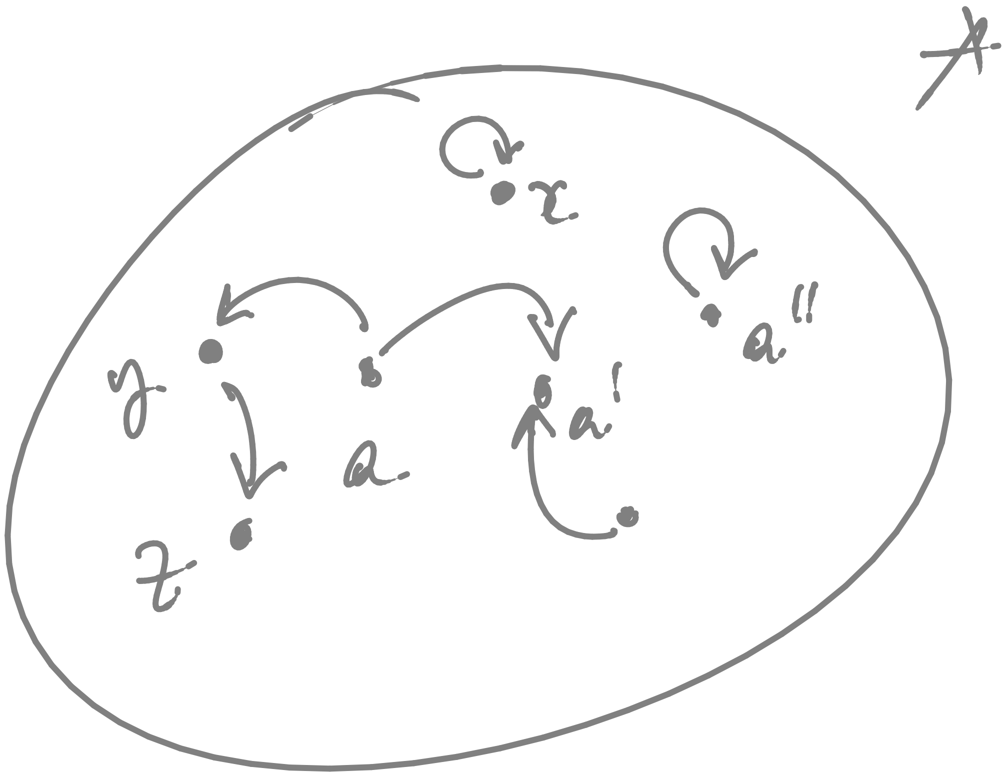
$$\underline{\text{mat}}(S) * \underline{\text{mat}}(R)$$

$\parallel$ $\underline{\text{claim}}$

$$\underline{\text{mat}}(S \circ R)$$

Directed graphs

Definition 108 *A* <u>directed graph</u> (A, R) *consists of a set* A *and a relation* R *on* A *(i.e. a relation from* A *to* A*).*



$a \longrightarrow a'$

iff $(a, a') \in R$

$(x, x) \in R^{\circ 0} = \mathrm{id}_A$

$(a, a') \in R^{\circ 1} = R$

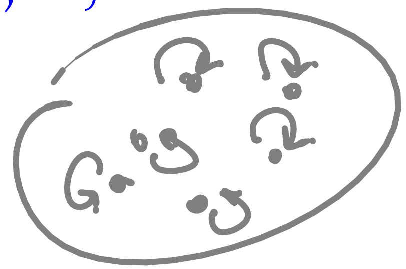
$(a, z) \in R^{\circ 2}$

$\underline{Rel}(A) = \mathcal{P}(A \times A) =$ the set of directed graphs on A.

Corollary 110 *For every set A, the structure*

$$(\,\mathrm{Rel}(A)\, ,\ \mathrm{id}_A\, ,\, \circ\,)$$

A



is a monoid.

Definition 111 *For $R \in \mathrm{Rel}(A)$ and $n \in \mathbb{N}$, we let*

$$R^{\circ n} \;=\; \underbrace{R \circ \cdots \circ R}_{n \text{ times}} \;\in \mathrm{Rel}(A)$$

be defined as id_A for $n = 0$, and as $R \circ R^{\circ m}$ for $n = m + 1$.

Paths

$R \subseteq A \times A$

Proposition 113 *Let* (A, R) *be a directed graph. For all* $n \in \mathbb{N}$ *and* $s, t \in A$, $s\ R^{\circ n}\ t$ *iff there exists a path of length* n *in* R *with source* s *and target* t.

PROOF: By induction on $n \in \mathbb{N}$.

Base case: $s\ R^{\circ 0}\ t \overset{?}{\iff} \exists$ a path of length 0 from s to t.

$\Updownarrow$ $\qquad\qquad\qquad\qquad$ $\Updownarrow$

$s = t$ $\qquad\qquad\qquad\qquad$ $s = t$

Ind step: Assume $\forall s, t.\ s\ R^{\circ n}\ t \iff \exists$ path of length n from s to t.

RTP: $\forall s, t$

$s\ R^{\circ (n+1)}\ t \iff \exists$ path of length $(n+1)$ from s to t.

$$s\, R^{\circ(n+1)}\, t \overset{\text{def}}{\iff} s\left(R \circ R^{\circ n}\right) t$$

$$\iff \exists\, u.\ s R u \wedge u R^{\circ n} t$$

$$\overset{\text{by IH}}{\iff} \exists\, u.\ s R u \wedge \exists \text{ path of length } n \text{ from } u \text{ to } t$$

$$\iff \exists \text{ path of length } (n+1) \text{ from } s \text{ to } t.$$

$\boxtimes$

Notation $\bigcup\{A_i \mid i \in I\} = \bigcup_{i\in I} A_i$

Definition 114 *For* $R \in \mathrm{Rel}(A)$, *let*

$$R^{\circ *} = \bigcup \{ R^{\circ n} \in \mathrm{Rel}(A) \mid n \in \mathbb{N} \} = \bigcup_{n\in\mathbb{N}} R^{\circ n} \quad .$$

Corollary 115 *Let* (A, R) *be a directed graph. For all* $s, t \in A$, $s \; R^{\circ *} \; t$ *iff there exists a path with source* s *and target* t *in* R.

$A = [n] = \{0, \ldots, n-1\}$ $\quad = \mathrm{rel}\big(\underbrace{\mathrm{mat}(R) * \cdots * \mathrm{mat}(R)}_{n \text{ times}}\big)$

How do we calculate $R^{\circ n}$?

The $(n \times n)$-matrix $M = \mathrm{mat}(R)$ of a finite directed graph $([n], R)$ for n a positive integer is called its <u>*adjacency matrix*</u>.

The adjacency matrix $M^* = \mathrm{mat}(R^{\circ *})$ can be computed by matrix multiplication and addition as M_n where

$$\begin{cases} M_0 &= I_n \\ M_{k+1} &= I_n + (M \cdot M_k) \end{cases}$$

This gives an algorithm for establishing or refuting the existence of paths in finite directed graphs.

Preorders

Definition 116 *A* preorder $(P, \sqsubseteq)$ *consists of a set* P *and a relation* $\sqsubseteq$ *on* P *(i.e.* $\sqsubseteq \in \mathcal{P}(P \times P)$*) satisfying the following two axioms.*

- *Reflexivity.*

$$\mathrm{id}_P \subseteq \sqsubseteq$$

$$\forall x \in P.\ x \sqsubseteq x$$

- *Transitivity.*

$$\forall x, y, z \in P.\ (x \sqsubseteq y \ \wedge\ y \sqsubseteq z) \implies x \sqsubseteq z$$

$$(\sqsubseteq \circ \sqsubseteq) \subseteq \sqsubseteq$$

partial orders $(P, \sqsubseteq)$

preorder s.t.

$x \sqsubseteq y \wedge y \sqsubseteq x$

$\Longrightarrow x = y$

Examples:

- $(\mathbb{R}, \leq)$ and $(\mathbb{R}, \geq)$.
- $(\mathcal{P}(A), \subseteq)$ and $(\mathcal{P}(A), \supseteq)$.
- $(\mathbb{Z}, \mid)$.

$n \mid n$

$n \mid m \wedge m \mid \ell \Rightarrow n \mid \ell$

$\bigcap \mathcal{F}_R$ is the least relation on A that contains R and it is a preorder.

Theorem 118 *For* $R \subseteq A \times A$, *let*

$$\mathcal{F}_R = \{ Q \subseteq A \times A \mid R \subseteq Q \wedge Q \textit{ is a preorder} \} \quad .$$

Then, (i) $R^{\circ *} \in \mathcal{F}_R$ *and* (ii) $R^{\circ *} \subseteq \bigcap \mathcal{F}_R$. *Hence,* $R^{\circ *} = \bigcap \mathcal{F}_R$.

PROOF: