Big intersections

Definition 92 *Let* $\mathcal{U}$ *be a set. For a collection of sets* $\mathcal{F} \subseteq \mathcal{P}(\mathcal{U})$*, we let the* big intersection *(relative to* $\mathcal{U}$*) be defined as*

$$\bigcap \mathcal{F} = \{ x \in \mathcal{U} \mid \forall A \in \mathcal{F}.\, x \in A \} \quad .$$

family of sets (subsets of $\mathcal{U}$)

intuitive idea

$$\mathcal{F} = \{ \cdots, A, B, \cdots, X, Y, \cdots \}$$

$$\bigcap \mathcal{F} = (\cdots \cap A \cap B \cap \cdots \cap X \cap Y \cap \cdots)$$

Fact

$$\forall A \in \mathcal{F}.\ \bigcap \mathcal{F} \subseteq A$$

$$X \subseteq \bigcap \mathcal{F} \iff \forall A \in \mathcal{F}.\, X \subseteq A$$

real numbers.

closure

Theorem 93 *Let*

$$\mathcal{F} = \Big\{ S \subseteq \mathbb{R} \;\Big|\; (0 \in S) \wedge (\forall x \in \mathbb{R}.\, x \in S \implies (x+1) \in S) \Big\} .$$

Then, (i) $\mathbb{N} \in \mathcal{F}$ *and* (ii) $\mathbb{N} \subseteq \bigcap \mathcal{F}$*. Hence,* $\bigcap \mathcal{F} = \mathbb{N}$*.*

PROOF:

Examples $\mathbb{Q} \in \mathcal{F}$

$\mathbb{R} \in \mathcal{F}$

$\mathbb{Z} \in \mathcal{F}$

$\bigcap \mathcal{F} \subseteq \mathbb{N}$

(ii) $\forall S \in \mathcal{F}.\ \mathbb{N} \subseteq S$

$\iff \forall S \in \mathcal{F}.\ \forall x \in \mathbb{N}.\ x \in S$

Proof by induction on $x \in \mathbb{N}$.

Example: Let A, B sets.

$\{A, B\}$

$\bigcup \{A, B\} = A \cup B$

Union axiom

$\boxed{\text{Every collection of sets has a union.}}$

$$\bigcup \mathcal{F}$$

$$x \in \bigcup \mathcal{F} \iff \exists X \in \mathcal{F}. x \in X$$

NB: To consider arbitrary big intersections one needs to check that the family of sets being intersected is non-empty.

For *non-empty* $\mathcal{F}$ we also have

$$\bigcap \mathcal{F}$$

defined by

$$\forall x.\ x \in \bigcap \mathcal{F} \iff (\forall X \in \mathcal{F}. x \in X) \quad .$$

$$\{1\} \times A = \{ (1, a) \mid a \in A \}$$

$$\{2\} \times B = \{ (2, b) \mid b \in B \}$$

Disjoint unions

Definition 94 *The* disjoint union $A \uplus B$ *of two sets* A *and* B *is the set*

$$A \uplus B \;=\; \big(\{1\} \times A\big) \cup \big(\{2\} \times B\big) \quad .$$

Thus,

$$\forall x.\, x \in (A \uplus B) \iff \big(\exists a \in A.\, x = (1, a)\big) \vee \big(\exists b \in B.\, x = (2, b)\big) \,.$$

<u>NB</u>: <u>datatype</u> (α, β) disjunion = one of α

$\mid$ two of β .

Proposition 96 *For all finite sets* A *and* B,

$$A \cap B = \emptyset \implies \#(A \cup B) = \#A + \#B \quad .$$

PROOF IDEA:

$\#A = n \qquad A = \{a_1, \ldots, a_n\}$

$\#B = m \qquad B = \{b_1 \ldots b_m\}$

$A \cup B = \{a_1 \ldots a_n, b_1 \ldots b_m\}$

$\#(A \cup B) = n + m.$

Corollary 97 *For all finite sets* A *and* B,

$$\#(A \uplus B) = \#A + \#B \quad .$$

Relations

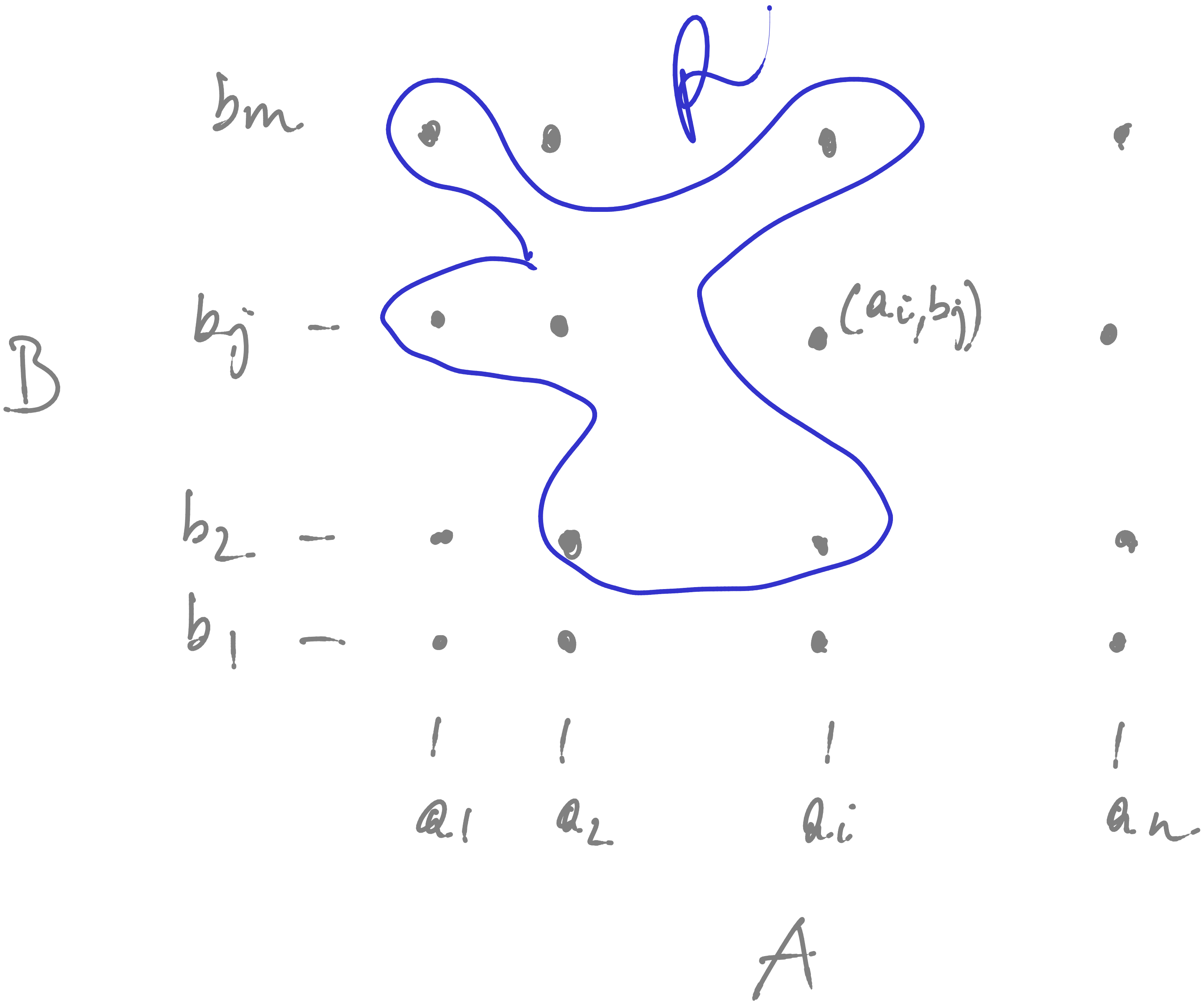
Definition 99 *A* <u>(binary) relation</u> R *from a set* A *to a set* B

$$R : A \longrightarrow\!\!\!\!\!\!+\; B \qquad \textit{or} \qquad R \in \mathrm{Rel}(A, B) \quad ,$$

is

$$R \subseteq A \times B \qquad \textit{or} \qquad R \in \mathcal{P}(A \times B) \quad .$$

Notation 100 *One typically writes* $a \, R \, b$ *for* $(a, b) \in R$*.*

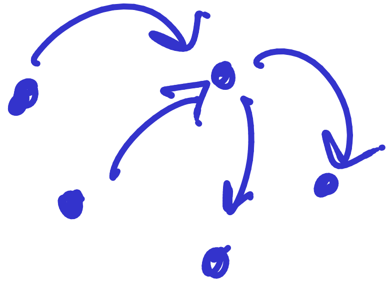


$$\#Rel(A,B) = \#P(A \times B) = 2^{\#(A \times B)} = 2^{\#(A) \cdot \#(B)}$$

$(1+2)\times 3 \to 3\times 3 \to 9$

Informal examples:

- Computation. $P \to P'$ $\quad e \to e'$
- Typing. $P : \tau$
- Program equivalence. $P \cong_{ctx} Q : \tau$
- Networks.



- Databases.

Relational

Notation $R : A \leftrightarrow B$ $a \in A, b \in B$

$(a, b) \in R \rightsquigarrow$ write aRb

Examples:

- Empty relation.

  $\emptyset : A \leftrightarrow B$ $(a \emptyset b \iff \mathbf{false})$

- Full relation.

  $(A \times B) : A \leftrightarrow B$ $(a\ (A \times B)\ b \iff \mathbf{true})$

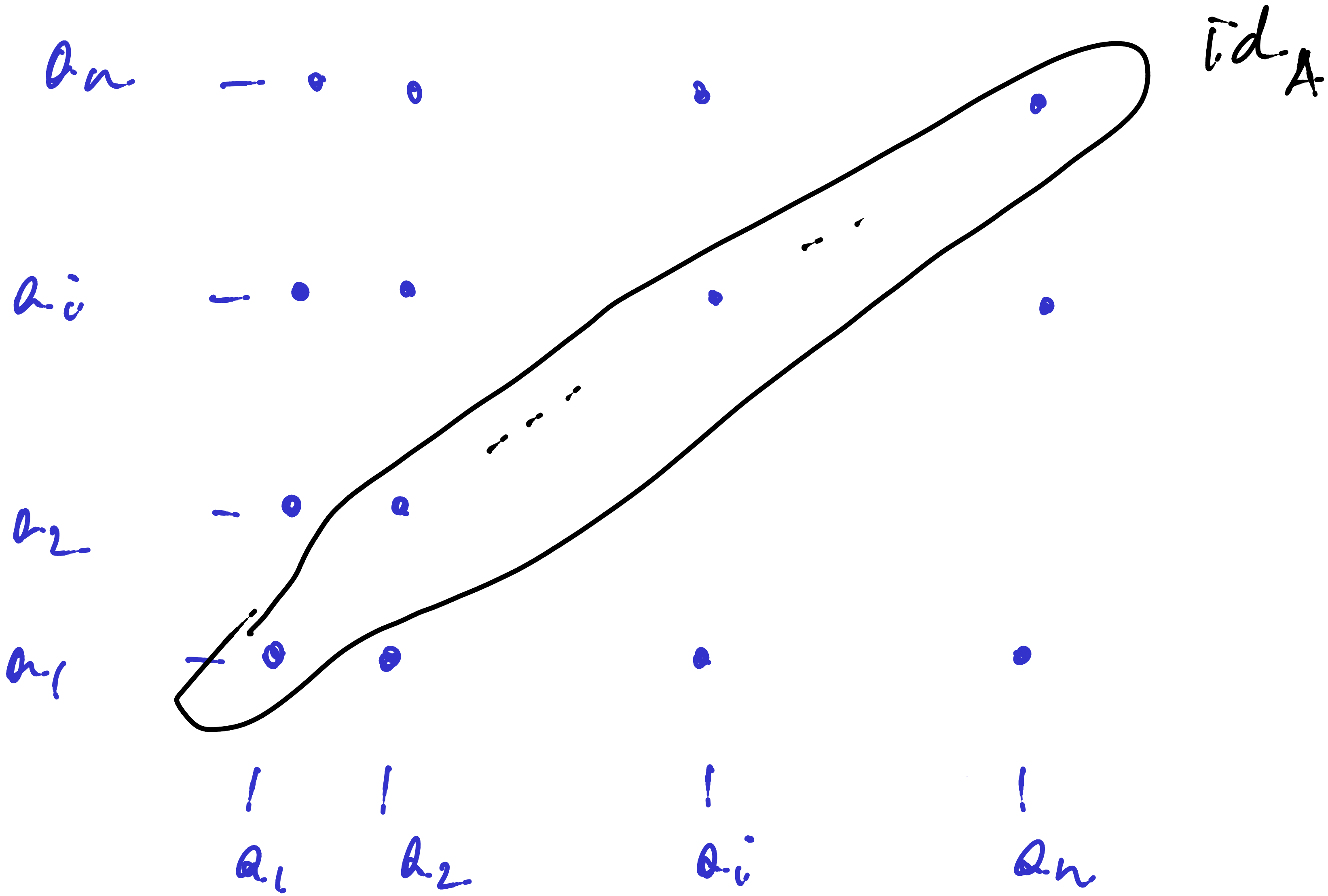
- Identity (or equality) relation.

  $\mathrm{id}_A = \{ (a, a) \mid a \in A \} : A \leftrightarrow A$ $(a\ \mathrm{id}_A\ a' \iff a = a')$

- Integer square root.

  $R_2 = \{ (m, n) \mid m = n^2 \} : \mathbb{N} \leftrightarrow \mathbb{Z}$ $(m\ R_2\ n \iff m = n^2)$

example: $4 R_2 2$, $4 R_2 -2$



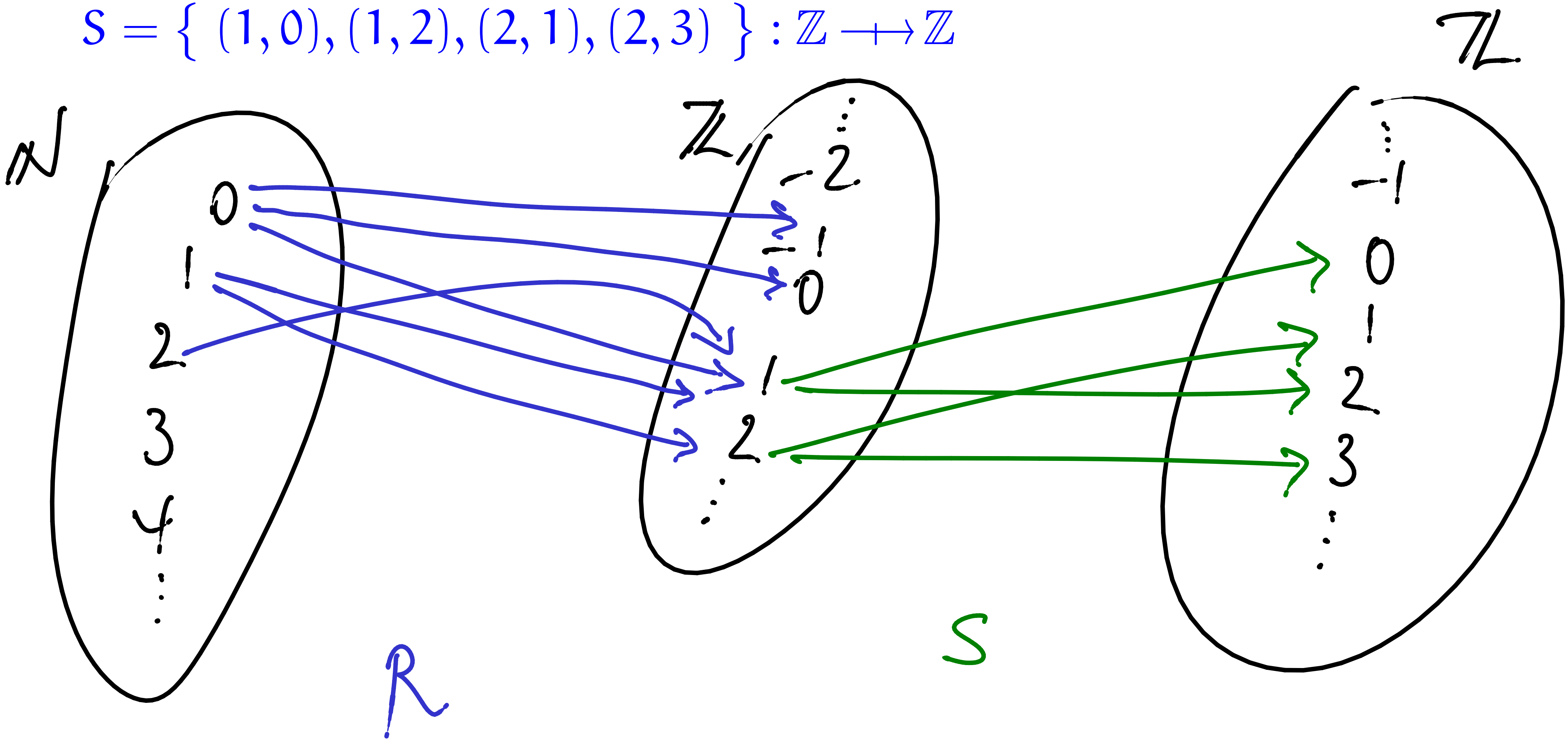
a_n
a_i
a_2
a_1
a_1 a_2 a_i a_n
id_A

Internal diagrams

Example:

$$R = \{ (0,0), (0,-1), (0,1), (1,2), (1,1), (2,1) \} : \mathbb{N} \rightarrowtail\!\!\!\!\!+ \mathbb{Z}$$

$$S = \{ (1,0), (1,2), (2,1), (2,3) \} : \mathbb{Z} \rightarrowtail\!\!\!\!\!+ \mathbb{Z}$$



$$\frac{R : A \nrightarrow B \qquad\qquad S : B \nrightarrow C}{S \circ R : A \nrightarrow C}$$

by def: $S \circ R \subseteq A \times C$

$$S \circ R = \{ (a, c) \in A \times C \mid \exists b \in B .\ aRb \wedge bSc \}$$

composition

Relational extensionality

$$R = S : A \longrightarrow\!\!\!\!\!\!+\;\; B$$

iff

$$\forall a \in A.\, \forall\, b \in B.\; a\, R\, b \iff a\, S\, b$$

W.l.o.g. one may write, abusing notation,

$T \circ S \circ R$

Theorem 102 *Relational composition is associative and has the identity relation as neutral element.*

- *Associativity.*

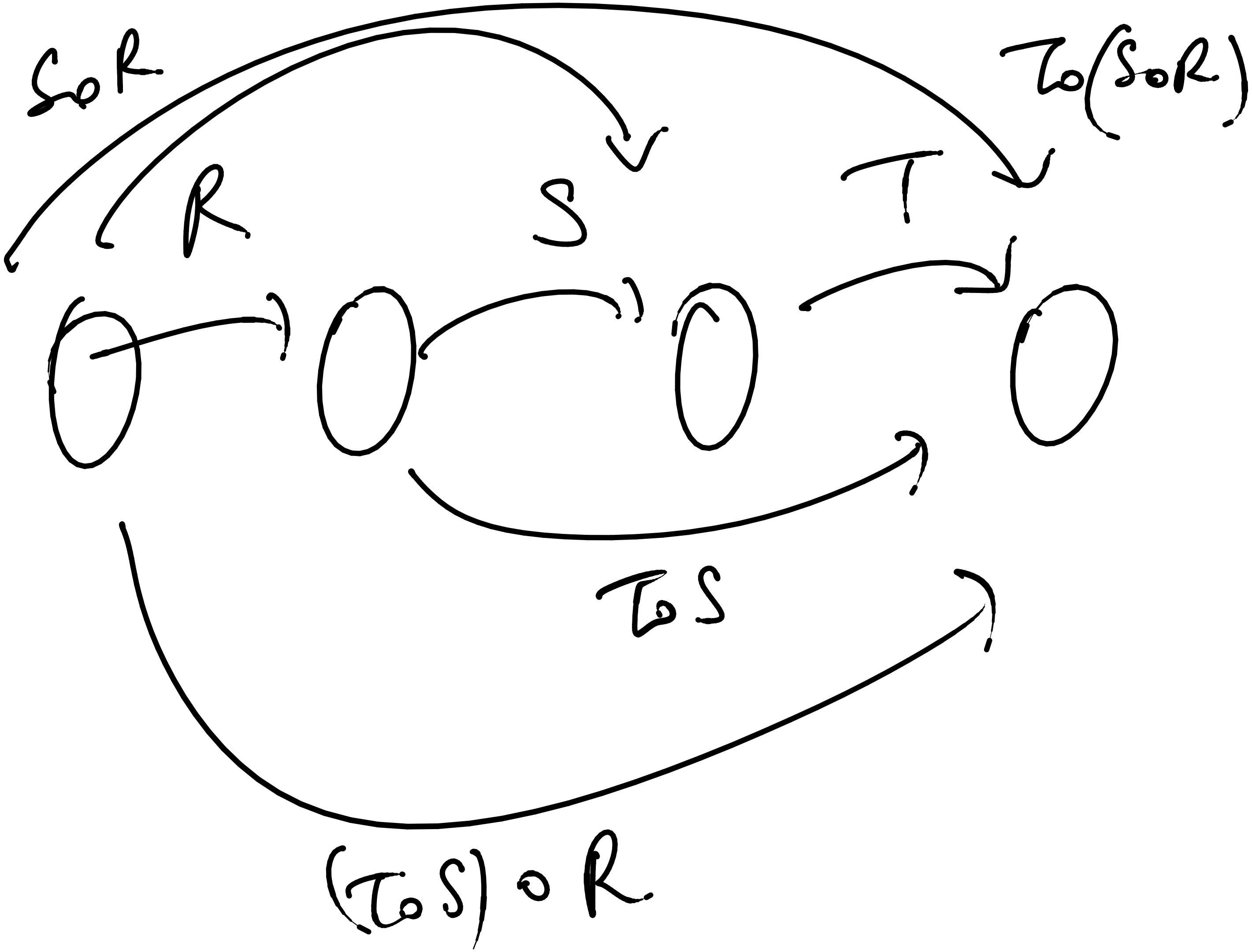
  For all $R : A \rightarrow\!\!\!\!\!+\, B$, $S : B \rightarrow\!\!\!\!\!+\, C$, *and* $T : C \rightarrow\!\!\!\!\!+\, D$,

$$(T \circ S) \circ R \;=\; T \circ (S \circ R)$$

- *Neutral element.*

  For all $R : A \rightarrow\!\!\!\!\!+\, B$,

$$R \circ \mathrm{id}_A \;=\; R \;=\; \mathrm{id}_B \circ R \quad .$$


S∘R
R
S
T
T∘(S∘R)
T∘S
(T∘S)∘R