$X \underbrace{\subseteq}_{\text{subset}} U \iff \forall x.\ x \in X \Rightarrow x \in U$

Powerset axiom

For any set, there is a set consisting of all its subsets.

$$\mathcal{P}(U)$$

$$\forall X.\ X \in \mathcal{P}(U) \iff X \subseteq U\ .$$

If U is finite of cardinality $n \in \mathbb{N}$ then $\mathcal{P}(U)$ is finite of cardinality 2^n.

The powerset Boolean algebra

$$(\ \mathcal{P}(U)\ ,\ \ \emptyset\ ,\ \ U\ ,\ \ \cup\ ,\ \ \cap\ ,\ \ (\cdot)^c\ )$$

For all $A, B \in \mathcal{P}(U)$,

$$\begin{aligned}
A \cup B &= \{x \in U \mid x \in A \vee x \in B\} && \in \mathcal{P}(U) \\
A \cap B &= \{x \in U \mid x \in A \wedge x \in B\} && \in \mathcal{P}(U) \\
A^c &= \{x \in U \mid \neg(x \in A)\} && \in \mathcal{P}(U)
\end{aligned}$$

Sets and logic

$\mathcal{P}(\mathcal{U})$	$\{\,\textbf{false}\,,\ \textbf{true}\,\}$
$\emptyset$	**false**
$\mathcal{U}$	**true**
$\cup$	$\vee$
$\cap$	$\wedge$
$(\cdot)^{\mathrm{c}}$	$\neg(\cdot)$

$\Longleftrightarrow$ $A, B \subseteq \mathcal{U}$.

Proposition 85 *Let* U *be a set and let* $A, B \in \mathcal{P}(\mathrm{U})$.

1. $\forall X \in \mathcal{P}(\mathrm{U}).\ A \cup B \subseteq X \iff (A \subseteq X \wedge B \subseteq X)$.

2. $\forall X \in \mathcal{P}(\mathrm{U}).\ X \subseteq A \cap B \iff (X \subseteq A \wedge X \subseteq B)$.

PROOF:

(1) $X \subseteq \mathcal{U}$ arbitrary

($\Longrightarrow$) Assume $A \cup B \subseteq X$

RTP: $A \subseteq X$ and $B \subseteq X$

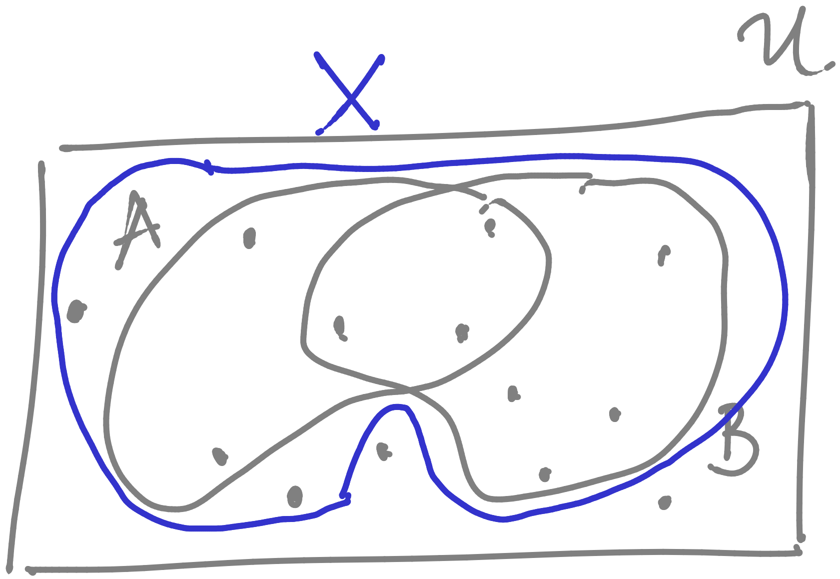
(i) $A \subseteq X$

Since $A \subseteq A \cup B$ and $A \cup B \subseteq X$

Then $A \subseteq X$

(ii) $B \subseteq X$

analogous ✓



Lemma

$A \subseteq A \cup B$

$B \subseteq A \cup B$

($\Leftarrow$) Assume $A \subseteq X$ and $B \subseteq X$

RTP: $A \cup B \subseteq X$

$\iff \forall x.\ x \in A \cup B \Rightarrow x \in X.$

Let x be arbitrary such that $x \in A \cup B$ $\iff$ $x \in A \vee x \in B$

RTP: $x \in X$.

Case $x \in A$: Then, since $A \subseteq X$, $x \in X$, and we are done.

Case $x \in B$: analogous.

$\boxtimes$

PROOF PRINCIPLE FOR UNIONS and INTERSECTIONS

Corollary 86 *Let* U *be a set and let* $A, B, C \in \mathcal{P}(\mathrm{U})$.

1. $$C = A \cup B$$

   iff

   $$[A \subseteq C \wedge B \subseteq C]$$

   $$\wedge$$

   $$[\forall X \in \mathcal{P}(\mathrm{U}).\ (A \subseteq X \ \wedge\ B \subseteq X) \implies C \subseteq X]$$

2. $$C = A \cap B$$

   iff

   $$[C \subseteq A \wedge C \subseteq B]$$

   $$\wedge$$

   $$[\forall X \in \mathcal{P}(\mathrm{U}).\ (X \subseteq A \ \wedge\ X \subseteq B) \implies X \subseteq C]$$

Pairing axiom

For every a and b, there is a set with a and b as its only elements.

$$\{ a , b \}$$

defined by

$$\forall x. \, x \in \{a, b\} \iff (x = a \lor x = b)$$

NB The set $\{a, a\}$ is abbreviated as $\{ a \}$, and referred to as a *singleton*.

$\forall x. \; x \in \{a, a\} \Leftrightarrow (x = a) \lor (x = a) \Leftrightarrow (x = a)$

$\underline{\text{NB}}$: $\# \emptyset = 0$

Examples:

- $\#\{\emptyset\} = 1$
- $\#\{\{\emptyset\}\} = 1$
- $\#\{\emptyset, \{\emptyset\}\} = 2$

$$\langle b, a \rangle = \{ \{b\}, \{a, b\} \}$$

Ordered pairing

For every pair a and b, the set

$$\{ \{a\}, \{a, b\} \}$$

is abbreviated as

$$\| $$

$$\langle a, b \rangle$$

and referred to as an *ordered pair*.

Proposition 87 (Fundamental property of ordered pairing)
For all a, b, x, y,

$$\langle a, b\rangle = \langle x, y\rangle \iff (a = x \wedge b = y) \quad .$$

PROOF: Let a, b, x, y be arbitrary.

($\Leftarrow$) Easy.

($\Rightarrow$) Assume $\{\{a\}, \{a, b\}\} = \{\{x\}, \{x, y\}\}$

<u>RTP</u>: $a = x$ and $b = y$

<u>Case</u> $a = b$: $\langle a, b\rangle = \{\{a\}\} = \{\{x\}, \{x, y\}\}$

$\Downarrow$ $\{a\} = \{x\}$ $\Downarrow$ $a = x$

$\Downarrow$ $\{a\} = \{x, y\}$ $\Downarrow$ $y = a$

<u>Case</u> $a \neq b$: Exercise!

Cartesian

Products

Cartesian plane

$\mathbb{R} \times \mathbb{R}$

The *product* $A \times B$ of two sets A and B is the set

$$A \times B = \{\, x \mid \exists\, a \in A, b \in B.\, x = (a, b) \,\}$$

$$= \{ (a,b) \mid a \in A \wedge b \in B \}$$

where

$$\forall a_1, a_2 \in A, b_1, b_2 \in B.$$
$$(a_1, b_1) = (a_2, b_2) \iff (a_1 = a_2 \wedge b_1 = b_2) \quad .$$

Thus,

$$\forall\, x \in A \times B.\, \exists!\, a \in A.\, \exists!\, b \in B.\, x = (a, b) \quad .$$

$$A = \{0, 1, 2\} \qquad B = \{a, b\}$$

$$A \times B = \{(0,a), (0,b), (1,a), (1,b), (2,a), (2,b)\}$$

$$\#A = 3$$

$$\#(A \times B) = 6 = \#A \cdot \#B$$

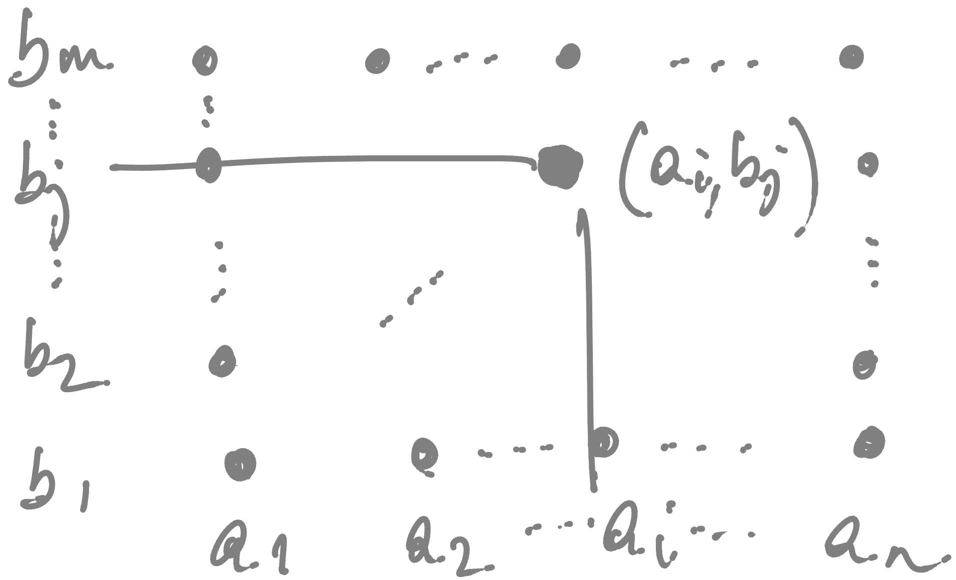
$$\#B = 2$$

Proposition 89 *For all finite sets* A *and* B,

$$\#(A \times B) = \#A \cdot \#B \quad .$$

PROOF IDEA: $\#A = n \qquad A = \{a_1, \dots, a_n\}$

$\#B = m \qquad B = \{b_1 \dots b_m\}$



$\mathcal{F}_2 = \{A, B\}$ $\quad$ $A, B \subseteq \mathcal{U}$

$\bigcup \mathcal{F}_2 = A \cup B$

$\mathcal{F}_3 = \{A, B, C\}$ $\quad$ $A, B, C \subseteq \mathcal{U}$

$\bigcup \mathcal{F}_3 = A \cup B \cup C$

$\mathcal{F}_0 = \{ \}$ $\quad$ $\mathcal{F}_1 = \{A\}$

$\bigcup \mathcal{F}_0 = \{ \}$ $\quad$ $\bigcup \mathcal{F}_1 = A$

given $\mathcal{F} \in \mathcal{P}(\mathcal{P}\,\mathcal{U})$ $\qquad$ $\mathcal{F} \subseteq \mathcal{P}(\mathcal{U})$

define $\bigcup \mathcal{F} \in \mathcal{P}(\mathcal{U})$ $\qquad$ $\bigcup \mathcal{F} \subseteq \mathcal{U}$

$$\bigcup \mathcal{F} = \{ x \in \mathcal{U} \mid \exists A \in \mathcal{F}.\ x \in A \}$$

$$\bigcup \{X, Y\} = \{ x \in \mathcal{U} \mid \exists A \in \{X, Y\}.\ x \in A \}$$

$$= \{ x \in \mathcal{U} \mid x \in X \vee x \in Y \} = X \cup Y$$

Big unions

Definition 90 *Let $\mathcal{U}$ be a set. For a collection of sets $\mathcal{F} \in \mathcal{P}(\mathcal{P}(\mathcal{U}))$, we let the* big union *(relative to $\mathcal{U}$) be defined as*

$$\bigcup \mathcal{F} = \{ x \in \mathcal{U} \mid \exists A \in \mathcal{F}.\, x \in A \} \in \mathcal{P}(\mathcal{U}) \quad .$$