Denotational Semantics

10 lectures for Part II CST 2019/20

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Course web page:
http://www.cl.cam.ac.uk/teaching/1920/DenotSem/
Topic 1

Introduction
What is this course about?

- General area.

  *Formal methods*: Mathematical techniques for the specification, development, and verification of software and hardware systems.

- Specific area.

  *Formal semantics*: Mathematical theories for ascribing meanings to computer languages.
Why do we care?
Why do we care?

• Rigour.
  
  ... specification of programming languages
  ... justification of program transformations
Why do we care?

- Rigour.
  - specification of programming languages
  - justification of program transformations

- Insight.
  - generalisations of notions computability
  - higher-order functions
  - data structures
• Feedback into language design.
  
  ... continuations
  
  ... monads
● Feedback into language design.
  
  \[ \ldots \text{continuations} \]
  
  \[ \ldots \text{monads} \]

● Reasoning principles.
  
  \[ \ldots \text{Scott induction} \]
  
  \[ \ldots \text{Logical relations} \]
  
  \[ \ldots \text{Co-induction} \]
Styles of formal semantics

Operational.

Axiomatic.

Denotational.
Styles of formal semantics

Operational.
Meanings for program phrases defined in terms of the steps of computation they can take during program execution.

Axiomatic.

Denotational.
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Meanings for program phrases defined indirectly via the axioms and rules of some logic of program properties.

Denotational.
Styles of formal semantics

Operational.
 Meanings for program phrases defined in terms of the steps of computation they can take during program execution.

Axiomatic.
 Meanings for program phrases defined indirectly via the axioms and rules of some logic of program properties.

Denotational.
 Concerned with giving mathematical models of programming languages. Meanings for program phrases defined abstractly as elements of some suitable mathematical structure.
Basic idea of denotational semantics

Syntax $\rightarrow$ Semantics

$P \mapsto [P]$
Basic idea of denotational semantics

Syntax $\xrightarrow{\mathcal{L}}$ Semantics

Recursive program $\mapsto$ Partial recursive function

\[ P \mapsto \llbracket P \rrbracket \]
Basic idea of denotational semantics

Syntax $\rightarrow$ Semantics

Recursive program $\mapsto$ Partial recursive function

Boolean circuit $\mapsto$ Boolean function

$P \mapsto \mathbb{[}P\mathbb{]}$
Basic idea of denotational semantics

Syntax \(\rightarrow\) Semantics

Recursive program \(\mapsto\) Partial recursive function

Boolean circuit \(\mapsto\) Boolean function

\[ P \mapsto \llbracket P \rrbracket \]

Concerns:

- Abstract models \(i.e.\) implementation/machine independent.
  \(\leadsto\) Lectures 2, 3 and 4.
Basic idea of denotational semantics

Syntax $\xrightarrow{\text{[ - ]}}$ Semantics

Recursive program $\mapsto$ Partial recursive function
Boolean circuit $\mapsto$ Boolean function

$P \mapsto \text{[} P \text{]}$

Concerns:

- Abstract models (i.e. implementation/machine independent).
  $\leadsto$ Lectures 2, 3 and 4.

- Compositionality.
  $\leadsto$ Lectures 5 and 6.
Basic idea of denotational semantics

Syntax $\rightarrow$ Semantics

Recursive program $\leftrightarrow$ Partial recursive function
Boolean circuit $\leftrightarrow$ Boolean function

$P \mapsto [P]$

Concerns:

- Abstract models (*i.e.* implementation/machine independent).
  $\leadsto$ Lectures 2, 3 and 4.

- Compositionality.
  $\leadsto$ Lectures 5 and 6.

- Relationship to computation (*e.g.* operational semantics).
  $\leadsto$ Lectures 7 and 8.
Characteristic features of a denotational semantics

- Each phrase (= part of a program), $P$, is given a denotation, $[[P]]$ — a mathematical object representing the contribution of $P$ to the meaning of any complete program in which it occurs.

- The denotation of a phrase is determined just by the denotations of its subphrases (one says that the semantics is compositional).
Basic example of denotational semantics (I)

IMP− syntax

Arithmetic expressions

\[ A \in A_{\text{exp}} ::= n \mid L \mid A + A \mid \ldots \]

where \( n \) ranges over integers and \( L \) over a specified set of locations \( \mathbb{L} \).

Boolean expressions

\[ B \in B_{\text{exp}} ::= \text{true} \mid \text{false} \mid A = A \mid \ldots \]
\[ \mid \neg B \mid \ldots \]

Commands

\[ C \in \text{Comm} ::= \text{skip} \mid L := A \mid C; C \]
\[ \mid \text{if } B \text{ then } C \text{ else } C \]
Basic example of denotational semantics (II)

Semantic functions

\[ A : \text{Aexp} \to (\text{State} \to \mathbb{Z}) \]

where

\[ \mathbb{Z} = \{ \ldots, -1, 0, 1, \ldots \} \]

\[ \text{State} = (\mathbb{L} \to \mathbb{Z}) \]
Basic example of denotational semantics (II)

Semantic functions

\[ A : \ A_{\text{exp}} \rightarrow (State \rightarrow \mathbb{Z}) \]
\[ B : \ B_{\text{exp}} \rightarrow (State \rightarrow \mathbb{B}) \]

where

\[ \mathbb{Z} = \{ \ldots, -1, 0, 1, \ldots \} \]
\[ \mathbb{B} = \{ \text{true}, \text{false} \} \]
\[ State = (\mathbb{L} \rightarrow \mathbb{Z}) \]
Basic example of denotational semantics (II)

Semantic functions

\[ A : \text{Aexp} \rightarrow (\text{State} \rightarrow \mathbb{Z}) \]
\[ B : \text{Bexp} \rightarrow (\text{State} \rightarrow \mathbb{B}) \]
\[ C : \text{Comm} \rightarrow (\text{State} \rightarrow \text{State}) \]

where

\[ \mathbb{Z} = \{ \ldots, -1, 0, 1, \ldots \} \]
\[ \mathbb{B} = \{ \text{true, false} \} \]
\[ \text{State} = (\mathbb{L} \rightarrow \mathbb{Z}) \]
Basic example of denotational semantics (III)

Semantic function $\mathcal{A}$

\[
\mathcal{A}[n] = \lambda s \in State. \; n
\]

\[
\mathcal{A}[L] = \lambda s \in State. \; s(L)
\]

\[
\mathcal{A}[A_1 + A_2] = \lambda s \in State. \; \mathcal{A}[A_1](s) + \mathcal{A}[A_2](s)
\]
Basic example of denotational semantics (IV)

Semantic function $B$

\[
B[\text{true}] = \lambda s \in \text{State}. \text{true}
\]

\[
B[\text{false}] = \lambda s \in \text{State}. \text{false}
\]

\[
B[A_1 = A_2] = \lambda s \in \text{State}. \text{eq}(A[A_1](s), A[A_2](s))
\]

where $\text{eq}(a, a') = \begin{cases} 
\text{true} & \text{if } a = a' \\
\text{false} & \text{if } a \neq a'
\end{cases}$
Basic example of denotational semantics (V)

Semantic function $C$

$$\llbracket \text{skip} \rrbracket = \lambda s \in \text{State}. \ s$$

**NB:** From now on the names of semantic functions are omitted!
A simple example of compositionality

Given partial functions $[C], [C'] : \text{State} \rightarrow \text{State}$ and a function $[B] : \text{State} \rightarrow \{\text{true}, \text{false}\}$, we can define

$$[\text{if } B \text{ then } C \text{ else } C'] =$$

$$\lambda s \in \text{State}. \text{if}( [B](s), [C](s), [C'])(s))$$

where

$$\text{if} (b, x, x') = \begin{cases} 
  x & \text{if } b = \text{true} \\
  x' & \text{if } b = \text{false} 
\end{cases}$$
Basic example of denotational semantics (VI)

Semantic function $C$

$$[L := A] = \lambda s \in \text{State.} \lambda \ell \in \mathbb{L}. \text{if } (\ell = L, [A](s), s(\ell))$$
Denotational semantics of sequential composition

Denotation of sequential composition $C; C'$ of two commands

$$\left[ C; C' \right] = \left[ C' \right] \circ \left[ C \right] = \lambda s \in \text{State}. \left[ C' \left( \left[ C \right] (s) \right) \right]$$

given by composition of the partial functions from states to states $\left[ C \right], \left[ C' \right] : \text{State} \rightarrow \text{State}$ which are the denotations of the commands.
Denotational semantics of sequential composition

Denotation of sequential composition $C; C'$ of two commands

$$[C; C'] = [C'] \circ [C] = \lambda s \in \text{State}. [C']([C](s))$$

given by composition of the partial functions from states to states $[C], [C'] : \text{State} \rightarrow \text{State}$ which are the denotations of the commands.

Cf. operational semantics of sequential composition:

$$C, s \Downarrow s' \quad C', s' \Downarrow s'' \quad \Rightarrow \quad C; C', s \Downarrow s''$$
\begin{itemize}
\item while $B$ do $C$
\end{itemize}
Fixed point property of \[ [\textbf{while } B \textbf{ do } C] \]

\[ [\textbf{while } B \textbf{ do } C] = f_{[B],[C]}([\textbf{while } B \textbf{ do } C]) \]

where, for each \( b : \text{State} \rightarrow \{\text{true, false}\} \) and \( c : \text{State} \rightarrow \text{State} \), we define

\( f_{b,c} : (\text{State} \rightarrow \text{State}) \rightarrow (\text{State} \rightarrow \text{State}) \)

as

\( f_{b,c} = \lambda w \in (\text{State} \rightarrow \text{State}). \lambda s \in \text{State}. \text{if} (b(s), w(c(s)), s). \)
Fixed point property of

\[[\text{while } B \text{ do } C]\]

\[[\text{while } B \text{ do } C] = f_{[B],[C]}([\text{while } B \text{ do } C])\]

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as

\[ f_{b,c} = \lambda w \in (\text{State} \rightarrow \text{State}). \lambda s \in \text{State}. \text{if } (b(s), w(c(s)), s). \]

- Why does \( w = f_{[B],[C]}(w) \) have a solution?
- What if it has several solutions—which one do we take to be \( [\text{while } B \text{ do } C] \)?
Approximating $\left[ \text{while } B \text{ do } C \right]$
Approximating $\textbf{while } B \textbf{ do } C$ \\

\[ f_{[B],[C]}^n(\bot) = \lambda s \in \text{State.} \]

\[
\begin{cases} 
[C]^k(s) & \text{if } \exists 0 \leq k < n. [B]([C]^k(s)) = \text{false} \\
& \text{and } \forall 0 \leq i < k. [B]([C]^i(s)) = \text{true} \\
\uparrow & \text{if } \forall 0 \leq i < n. [B]([C]^i(s)) = \text{true}
\end{cases}
\]
\[ D \overset{\text{def}}{=} (State \rightarrow State) \]

- **Partial order** \( \sqsubseteq \) on \( D \):
  
  \( w \sqsubseteq w' \) iff for all \( s \in State \), if \( w \) is defined at \( s \) then so is \( w' \) and moreover \( w(s) = w'(s) \).
  
  iff the graph of \( w \) is included in the graph of \( w' \).

- **Least element** \( \perp \in D \) w.r.t. \( \sqsubseteq \):
  
  \( \perp \) = totally undefined partial function
  = partial function with empty graph
  (satisfies \( \perp \sqsubseteq w \), for all \( w \in D \)).
Topic 2

Least Fixed Points
All domains of computation are partial orders with a least element.
Thesis

All domains of computation are partial orders with a least element.

All computable functions are monotonic.
Partially ordered sets

A binary relation $\sqsubseteq$ on a set $D$ is a partial order iff it is

reflexive: $\forall d \in D. d \sqsubseteq d$

transitive: $\forall d, d', d'' \in D. d \sqsubseteq d' \sqsubseteq d'' \Rightarrow d \sqsubseteq d''$

anti-symmetric: $\forall d, d' \in D. d \sqsubseteq d' \sqsubseteq d \Rightarrow d = d'$.

Such a pair $(D, \sqsubseteq)$ is called a partially ordered set, or poset.
\[ x \sqsubseteq x \]

\[ x \sqsubseteq y \quad y \sqsubseteq z \]
\[ x \sqsubseteq z \]

\[ x \sqsubseteq y \quad y \sqsubseteq x \]
\[ x = y \]
Domain of partial functions, $X \rightarrow Y$
Domain of partial functions, $X \twoheadrightarrow Y$

**Underlying set:** all partial functions, $f$, with domain of definition $\text{dom}(f) \subseteq X$ and taking values in $Y$. 
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**Underlying set:** all partial functions, $f$, with domain of definition $\text{dom}(f) \subseteq X$ and taking values in $Y$.

**Partial order:**

$f \sqsubseteq g \iff \text{dom}(f) \subseteq \text{dom}(g)$ and $\forall x \in \text{dom}(f). f(x) = g(x)$

iff $\text{graph}(f) \subseteq \text{graph}(g)$
Monotonicity

• A function $f : D \rightarrow E$ between posets is monotone iff

$$\forall d, d' \in D. \ d \sqsubseteq d' \Rightarrow f(d) \sqsubseteq f(d').$$

\[
\frac{x \sqsubseteq y}{f(x) \sqsubseteq f(y)} \quad (f \text{ monotone})
\]
Least Elements

Suppose that $D$ is a poset and that $S$ is a subset of $D$. An element $d \in S$ is the least element of $S$ if it satisfies

$$\forall x \in S. \ d \sqsubseteq x .$$

• Note that because $\sqsubseteq$ is anti-symmetric, $S$ has at most one least element.

• Note also that a poset may not have least element.
Pre-fixed points

Let $D$ be a poset and $f : D \to D$ be a function.

An element $d \in D$ is a pre-fixed point of $f$ if it satisfies $f(d) \sqsubseteq d$.

The least pre-fixed point of $f$, if it exists, will be written $\text{fix}(f)$.

It is thus (uniquely) specified by the two properties:

\begin{align*}
\text{lfp1} \quad & f(\text{fix}(f)) \sqsubseteq \text{fix}(f) \\
\text{lfp2} \quad & \forall d \in D. \ f(d) \sqsubseteq d \Rightarrow \text{fix}(f) \sqsubseteq d.
\end{align*}
2. Let $D$ be a poset and let $f : D \to D$ be a function with a least pre-fixed point $\text{fix}(f) \in D$. For all $x \in D$, to prove that $\text{fix}(f) \sqsubseteq x$ it is enough to establish that $f(x) \sqsubseteq x$. 
2. Let $D$ be a poset and let $f : D \to D$ be a function with a least pre-fixed point $\text{fix}(f) \in D$.

For all $x \in D$, to prove that $\text{fix}(f) \sqsubseteq x$ it is enough to establish that $f(x) \sqsubseteq x$.

\[
\frac{f(x) \sqsubseteq x}{\text{fix}(f) \sqsubseteq x}
\]
Proof principle

1.

\[
f(f(x)) \sqsubseteq f(x)
\]

2. Let \( D \) be a poset and let \( f : D \to D \) be a function with a least pre-fixed point \( \text{fix}(f) \in D \).

For all \( x \in D \), to prove that \( \text{fix}(f) \sqsubseteq x \) it is enough to establish that \( f(x) \sqsubseteq x \).

\[
\frac{f(x) \sqsubseteq x}{\text{fix}(f) \sqsubseteq x}
\]
Least pre-fixed points are fixed points

If it exists, the least pre-fixed point of a mononote function on a partial order is necessarily a fixed point.
All domains of computation are complete partial orders with a least element.
All domains of computation are complete partial orders with a least element.

All computable functions are continuous.
A chain complete poset, or \textit{cpo} for short, is a poset \((D, \sqsubseteq)\) in which all countable increasing chains \(d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \ldots\) have least upper bounds, \(\bigsqcup_{n \geq 0} d_n:\)

\[
\forall m \geq 0 . \ d_m \sqsubseteq \bigsqcup_{n \geq 0} d_n \tag{\text{lub1}}
\]

\[
\forall d \in D . (\forall m \geq 0 . \ d_m \sqsubseteq d) \Rightarrow \bigsqcup_{n \geq 0} d_n \sqsubseteq d. \tag{\text{lub2}}
\]

A \textit{domain} is a cpo that possesses a least element, \(\bot:\)

\[
\forall d \in D . \ \bot \sqsubseteq d.
\]
\[
\forall n \geq 0 . \ x_n \sqsubseteq x \\
\sqcup_{n \geq 0} x_n \sqsubseteq x
\]

\[
\text{〈} x_i \text{〉 a chain}
\]

\[
\exists i \geq 0 . \ x_i \sqsubseteq \sqcup_{n \geq 0} x_n
\]

\[
\text{〈} x_n \text{〉 a chain}
\]
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iff $\text{graph}(f) \subseteq \text{graph}(g)$
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$f \sqsubseteq g$ iff $\text{dom}(f) \subseteq \text{dom}(g)$ and

$\forall x \in \text{dom}(f). \ f(x) = g(x)$

iff $\text{graph}(f) \subseteq \text{graph}(g)$

**Lub of chain** $f_0 \sqsubseteq f_1 \sqsubseteq f_2 \sqsubseteq \ldots$ is the partial function $f$ with

$\text{dom}(f) = \bigcup_{n \geq 0} \text{dom}(f_n)$ and

$$f(x) = \begin{cases} f_n(x) & \text{if } x \in \text{dom}(f_n), \text{ some } n \\ \text{undefined} & \text{otherwise} \end{cases}$$
Domain of partial functions, $X \rightarrow Y$

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**Partial order:**

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**Lub of chain** $f_0 \sqsubseteq f_1 \sqsubseteq f_2 \sqsubseteq \ldots$ is the partial function $f$ with $\text{dom}(f) = \bigcup_{n \geq 0} \text{dom}(f_n)$ and

$$f(x) = \begin{cases} f_n(x) & \text{if } x \in \text{dom}(f_n), \text{ some } n \\ \text{undefined} & \text{otherwise} \end{cases}$$

**Least element** $\bot$ is the totally undefined partial function.
Some properties of lubs of chains

Let $D$ be a cpo.

1. For $d \in D$, $\bigsqcup_n d = d$.

2. For every chain $d_0 \sqsubseteq d_1 \sqsubseteq \ldots \sqsubseteq d_n \sqsubseteq \ldots$ in $D$,

   $$\bigsqcup_n d_n = \bigsqcup_n d_{N+n}$$

   for all $N \in \mathbb{N}$.
3. For every pair of chains $d_0 \subseteq d_1 \subseteq \ldots \subseteq d_n \subseteq \ldots$ and $e_0 \subseteq e_1 \subseteq \ldots \subseteq e_n \subseteq \ldots$ in $D$, if $d_n \subseteq e_n$ for all $n \in \mathbb{N}$ then $\bigcup_n d_n \subseteq \bigcup_n e_n$. 
3. For every pair of chains \( d_0 \sqsubseteq d_1 \sqsubseteq \ldots \sqsubseteq d_n \sqsubseteq \ldots \) and \( e_0 \sqsubseteq e_1 \sqsubseteq \ldots \sqsubseteq e_n \sqsubseteq \ldots \) in \( D \),

if \( d_n \sqsubseteq e_n \) for all \( n \in \mathbb{N} \) then \( \bigcup_n d_n \sqsubseteq \bigcup_n e_n \).

\[
\forall n \geq 0. \ x_n \sqsubseteq y_n \quad (\langle x_n \rangle \text{ and } \langle y_n \rangle \text{ chains})
\]

\[
\bigcup_n x_n \sqsubseteq \bigcup_n y_n
\]
Diagonalising a double chain

Lemma. Let $D$ be a cpo. Suppose that the doubly-indexed family of elements $d_{m,n} \in D$ ($m, n \geq 0$) satisfies

$$m \leq m' \& n \leq n' \Rightarrow d_{m,n} \sqsubseteq d_{m',n'}.$$  \hfill (†)

Then

$$\bigsqcup_{n \geq 0} d_{0,n} \sqsubseteq \bigsqcup_{n \geq 0} d_{1,n} \sqsubseteq \bigsqcup_{n \geq 0} d_{2,n} \sqsubseteq \ldots$$

and

$$\bigsqcup_{m \geq 0} d_{m,0} \sqsubseteq \bigsqcup_{m \geq 0} d_{m,1} \sqsubseteq \bigsqcup_{m \geq 0} d_{m,3} \sqsubseteq \ldots$$
Diagonalising a double chain

**Lemma.** Let $D$ be a cpo. Suppose that the doubly-indexed family of elements $d_{m,n} \in D$ $(m, n \geq 0)$ satisfies

$$m \leq m' \ & \ n \leq n' \ \Rightarrow \ d_{m,n} \sqsubseteq d_{m',n'}.$$  

Then

$$\bigsqcup_{n \geq 0} d_{0,n} \sqsubseteq \bigsqcup_{n \geq 0} d_{1,n} \sqsubseteq \bigsqcup_{n \geq 0} d_{2,n} \sqsubseteq \ldots$$

and

$$\bigsqcup_{m \geq 0} d_{m,0} \sqsubseteq \bigsqcup_{m \geq 0} d_{m,1} \sqsubseteq \bigsqcup_{m \geq 0} d_{m,3} \sqsubseteq \ldots$$

Moreover

$$\bigsqcup_{m \geq 0} \left( \bigsqcup_{n \geq 0} d_{m,n} \right) = \bigsqcup_{k \geq 0} d_{k,k} = \bigsqcup_{n \geq 0} \left( \bigsqcup_{m \geq 0} d_{m,n} \right).$$
Continuity and strictness

- If $D$ and $E$ are cpo’s, the function $f$ is continuous iff
  1. it is monotone, and
  2. it preserves lubs of chains, i.e. for all chains $d_0 \sqsubseteq d_1 \sqsubseteq \ldots$ in $D$, it is the case that
     
     $$f\left(\bigsqcup_{n \geq 0} d_n\right) = \bigsqcup_{n \geq 0} f\left(d_n\right) \text{ in } E.$$
Continuity and strictness

- If $D$ and $E$ are cpo’s, the function $f$ is continuous iff
  1. it is monotone, and
  2. it preserves lubs of chains, i.e. for all chains $d_0 \sqsubseteq d_1 \sqsubseteq \ldots$ in $D$, it is the case that

$$f(\bigsqcup_{n \geq 0} d_n) = \bigsqcup_{n \geq 0} f(d_n) \quad \text{in } E.$$  

- If $D$ and $E$ have least elements, then the function $f$ is strict iff $f(\bot) = \bot$.  

Tarski’s Fixed Point Theorem

Let \( f : D \rightarrow D \) be a continuous function on a domain \( D \). Then

- \( f \) possesses a least pre-fixed point, given by
  \[
  \text{fix}(f) = \bigsqcup_{n \geq 0} f^n(\perp).
  \]

- Moreover, \( \text{fix}(f) \) is a fixed point of \( f \), i.e. satisfies
  \[
  f(\text{fix}(f)) = \text{fix}(f),
  \]
  and hence is the least fixed point of \( f \).
\[ [\text{while } B \text{ do } C] \]

\[
[\text{while } B \text{ do } C] = \text{fix}(f_{[B],[C]})
\]

\[
= \bigcup_{n \geq 0} f_{[B],[C]}^n(\bot)
\]

\[
= \lambda s \in \text{State.}
\]

\[
\begin{cases} 
[\text{C}]^k(s) & \text{if } k \geq 0 \text{ is such that } [B](\text{[C]}^k(s)) = \text{false} \\
& \text{and } [B](\text{[C]}^i(s)) = \text{true} \text{ for all } 0 \leq i < k \\
\text{undefined} & \text{if } [B](\text{[C]}^i(s)) = \text{true} \text{ for all } i \geq 0 
\end{cases}
\]
Topic 3
Constructions on Domains
Discrete cpo’s and flat domains

For any set $X$, the relation of equality

$$x \sqsubseteq x' \overset{\text{def}}{\iff} x = x' \quad (x, x' \in X)$$

makes $(X, \sqsubseteq)$ into a cpo, called the discrete cpo with underlying set $X$. 
Discrete cpo’s and flat domains

For any set $X$, the relation of equality

\[ x \sqsubseteq x' \iff x = x' \quad (x, x' \in X) \]

makes $(X, \sqsubseteq)$ into a cpo, called the discrete cpo with underlying set $X$.

Let $X_\bot \overset{\text{def}}{=} X \cup \{\bot\}$, where $\bot$ is some element not in $X$. Then

\[ d \sqsubseteq d' \overset{\text{def}}{=} (d = d') \lor (d = \bot) \quad (d, d' \in X_\bot) \]

makes $(X_\bot, \sqsubseteq)$ into a domain (with least element $\bot$), called the flat domain determined by $X$. 
Binary product of cpo’s and domains

The product of two cpo’s \((D_1, \sqsubseteq_1)\) and \((D_2, \sqsubseteq_2)\) has underlying set

\[
D_1 \times D_2 = \{(d_1, d_2) \mid d_1 \in D_1 \& d_2 \in D_2\}
\]

and partial order \(\sqsubseteq\) defined by

\[
(d_1, d_2) \sqsubseteq (d'_1, d'_2) \iff d_1 \sqsubseteq_1 d'_1 \& d_2 \sqsubseteq_2 d'_2.
\]

\[
(x_1, x_2) \sqsubseteq (y_1, y_2) \quad \underbrace{x_1 \sqsubseteq_1 y_1 \quad x_2 \sqsubseteq_2 y_2}_{\text{in \(D_1 \times D_2\)}}
\]
Lubs of chains are calculated componentwise:

\[
\bigcup_{n \geq 0} (d_{1,n}, d_{2,n}) = (\bigcup_{i \geq 0} d_{1,i}, \bigcup_{j \geq 0} d_{2,j}) .
\]

If \((D_1, \sqsubseteq_1)\) and \((D_2, \sqsubseteq_2)\) are domains so is \((D_1 \times D_2, \sqsubseteq)\) and \(\perp_{D_1 \times D_2} = (\perp_{D_1}, \perp_{D_2})\).
Continuous functions of two arguments

**Proposition.** Let $D, E, F$ be cpo’s. A function $f : (D \times E) \rightarrow F$ is monotone if and only if it is monotone in each argument separately:

\[
\forall d, d' \in D, e \in E. \ d \sqsubseteq d' \implies f(d, e) \sqsubseteq f(d', e)
\]

\[
\forall d \in D, e, e' \in E. \ e \sqsubseteq e' \implies f(d, e) \sqsubseteq f(d, e').
\]

Moreover, it is continuous if and only if it preserves lubs of chains in each argument separately:

\[
f(\bigsqcup_{m \geq 0} d_m, e) = \bigsqcup_{m \geq 0} f(d_m, e)
\]

\[
f(d, \bigsqcup_{n \geq 0} e_n) = \bigsqcup_{n \geq 0} f(d, e_n).
\]
• A couple of derived rules:

\[
\frac{x \sqsubseteq x' \quad y \sqsubseteq y'}{f(x, y) \sqsubseteq f(x', y')} \quad (f \text{ monotone})
\]

\[
f(\bigsqcup_m x_m, \bigsqcup_n y_n) = \bigsqcup_k f(x_k, y_k)
\]
Given cpo’s \((D, \sqsubseteq_D)\) and \((E, \sqsubseteq_E)\), the function cpo \((D \rightarrow E, \sqsubseteq)\) has underlying set

\[(D \rightarrow E) \overset{\text{def}}{=} \{ f \mid f : D \rightarrow E \text{ is a continuous function} \}\]

and partial order: \(f \sqsubseteq f' \overset{\text{def}}{\iff} \forall d \in D . f(d) \sqsubseteq_E f'(d)\).
Function cpo’s and domains

Given cpo’s \((D, \sqsubseteq_D)\) and \((E, \sqsubseteq_E)\), the function cpo \((D \to E, \sqsubseteq)\) has underlying set

\[
(D \to E) \overset{\text{def}}{=} \{ f \mid f : D \to E \text{ is a continuous function} \}
\]

and partial order: \(f \sqsubseteq_S f' \overset{\text{def}}{\iff} \forall d \in D . f(d) \sqsubseteq_E f'(d)\).

- A derived rule:

\[
\frac{f \sqsubseteq (D \to E) \quad g \quad x \sqsubseteq_D y}{f(x) \sqsubseteq g(y)}
\]
Lubs of chains are calculated ‘argumentwise’ (using lubs in $E$):

$$\bigsqcup_{n \geq 0} f_n = \lambda d \in D. \bigsqcup_{n \geq 0} f_n(d) .$$

If $E$ is a domain, then so is $D \rightarrow E$ and $\perp_{D \rightarrow E}(d) = \perp_E$, all $d \in D$. 
Lubs of chains are calculated ‘argumentwise’ (using lubs in $E$):

\[
\bigsqcup_{n \geq 0} f_n = \lambda d \in D. \ \bigsqcup_{n \geq 0} f_n(d).
\]

- A derived rule:

\[
(\bigsqcup_n f_n)(\bigsqcup_m x_m) = \bigsqcup_k f_k(x_k)
\]

If $E$ is a domain, then so is $D \to E$ and $\bot_{D \to E}(d) = \bot_E$, all $d \in D$. 

Continuity of composition

For cpo's $D, E, F$, the composition function

$$\circ : ((E \to F) \times (D \to E)) \to (D \to F)$$

defined by setting, for all $f \in (D \to E)$ and $g \in (E \to F)$,

$$g \circ f = \lambda d \in D. g(f(d))$$

is continuous.
Continuity of the fixpoint operator

Let $D$ be a domain.

By Tarski's Fixed Point Theorem we know that each continuous function $f \in (D \rightarrow D)$ possesses a least fixed point, $\text{fix}(f) \in D$.

**Proposition.** The function

$$\text{fix} : (D \rightarrow D) \rightarrow D$$

is continuous.
Topic 4

Scott Induction
Scott’s Fixed Point Induction Principle

Let \( f : D \to D \) be a continuous function on a domain \( D \).

For any admissible subset \( S \subseteq D \), to prove that the least fixed point of \( f \) is in \( S \), i.e. that

\[
\text{fix}(f) \in S ,
\]

it suffices to prove

\[
\forall d \in D \ (d \in S \implies f(d) \in S) .
\]
Chain-closed and admissible subsets

Let $D$ be a cpo. A subset $S \subseteq D$ is called \textit{chain-closed} iff for all chains $d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \ldots$ in $D$

$$(\forall n \geq 0 . d_n \in S) \Rightarrow \left( \bigsqcup_{n \geq 0} d_n \right) \in S$$

If $D$ is a domain, $S \subseteq D$ is called \textit{admissible} iff it is a chain-closed subset of $D$ and $\bot \in S$. 
Chain-closed and admissible subsets

Let $D$ be a cpo. A subset $S \subseteq D$ is called **chain-closed** iff for all chains $d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \ldots$ in $D$

$$\left( \forall n \geq 0 . d_n \in S \right) \Rightarrow \left( \bigsqcup_{n \geq 0} d_n \right) \in S$$

If $D$ is a domain, $S \subseteq D$ is called **admissible** iff it is a chain-closed subset of $D$ and $\bot \in S$.

A property $\Phi(d)$ of elements $d \in D$ is called **chain-closed** (resp. **admissible**) iff $\{ d \in D \mid \Phi(d) \}$ is a chain-closed (resp. admissible) subset of $D$. 
Let $D, E$ be cpos.

**Basic relations:**

- For every $d \in D$, the subset

  $$\downarrow(d) \overset{\text{def}}{=} \{ x \in D \mid x \sqsubseteq d \}$$

  of $D$ is chain-closed.
Let $D, E$ be cpos.

**Basic relations:**

- For every $d \in D$, the subset 
  \[ \downarrow(d) \overset{\text{def}}{=} \{ x \in D \mid x \sqsubseteq d \} \]
  of $D$ is chain-closed.

- The subsets 
  \[ \{(x, y) \in D \times D \mid x \sqsubseteq y\} \]

  and 
  \[ \{(x, y) \in D \times D \mid x = y\} \]

  of $D \times D$ are chain-closed.
Example (I): Least pre-fixed point property

Let $D$ be a domain and let $f : D \to D$ be a continuous function.

$$\forall d \in D. f(d) \sqsubseteq d \implies \text{fix}(f) \sqsubseteq d$$
Example (I): Least pre-fixed point property

Let $D$ be a domain and let $f : D \rightarrow D$ be a continuous function.

$$\forall d \in D. f(d) \sqsubseteq d \implies \text{fix}(f) \sqsubseteq d$$

Proof by Scott induction.

Let $d \in D$ be a pre-fixed point of $f$. Then,

$$x \in \downarrow(d) \implies x \sqsubseteq d$$

$$\implies f(x) \sqsubseteq f(d)$$

$$\implies f(x) \sqsubseteq d$$

$$\implies f(x) \in \downarrow(d)$$

Hence,

$$\text{fix}(f) \in \downarrow(d).$$
Inverse image:

Let \( f : D \to E \) be a continuous function.

If \( S \) is a chain-closed subset of \( E \) then the inverse image

\[
  f^{-1}S = \{ x \in D \mid f(x) \in S \}
\]

is an chain-closed subset of \( D \).
Example (II)

Let $D$ be a domain and let $f, g : D \to D$ be continuous functions such that $f \circ g \sqsubseteq g \circ f$. Then,

$$f(\bot) \sqsubseteq g(\bot) \implies \text{fix}(f) \sqsubseteq \text{fix}(g).$$
Example (II)

Let $D$ be a domain and let $f, g : D \to D$ be continuous functions such that $f \circ g \sqsubseteq g \circ f$. Then,

$$f(\bot) \sqsubseteq g(\bot) \implies \text{fix}(f) \sqsubseteq \text{fix}(g).$$

**Proof by Scott induction.**

Consider the admissible property $\Phi(x) \equiv (f(x) \sqsubseteq g(x))$ of $D$.

Since

$$f(x) \sqsubseteq g(x) \implies g(f(x)) \sqsubseteq g(g(x)) \implies f(g(x)) \sqsubseteq g(g(x))$$

we have that

$$f(\text{fix}(g)) \sqsubseteq g(\text{fix}(g)).$$
Logical operations:

- If $S, T \subseteq D$ are chain-closed subsets of $D$ then $S \cup T$ and $S \cap T$ are chain-closed subsets of $D$.

- If $\{S_i\}_{i \in I}$ is a family of chain-closed subsets of $D$ indexed by a set $I$, then $\bigcap_{i \in I} S_i$ is a chain-closed subset of $D$.

- If a property $P(x, y)$ determines a chain-closed subset of $D \times E$, then the property $\forall x \in D. P(x, y)$ determines a chain-closed subset of $E$. **Building chain-closed subsets (III)**
Example (III): Partial correctness

Let $F : \text{State} \rightarrow \text{State}$ be the denotation of

$$\text{while } X > 0 \text{ do } (Y := X \times Y; X := X - 1).$$

For all $x, y \geq 0$,

$F[X \mapsto x, Y \mapsto y] \downarrow$

$\implies F[X \mapsto x, Y \mapsto y] = [X \mapsto 0, Y \mapsto !x \cdot y].$
Recall that

\[ \mathcal{F} = \text{fix}(f) \]

where \( f : (\text{State} \to \text{State}) \to (\text{State} \to \text{State}) \) is given by

\[
f(w) = \lambda(x, y) \in \text{State}. \begin{cases} 
  (x, y) & \text{if } x \leq 0 \\
  w(x - 1, x \cdot y) & \text{if } x > 0
\end{cases}
\]
**Proof by Scott induction.**

We consider the admissible subset of \((\text{State} \rightarrow \text{State})\) given by

\[
S = \left\{ w \right\} \quad \forall x, y \geq 0. \\
\quad w[X \mapsto x, Y \mapsto y] \downarrow \\
\quad \Rightarrow w[X \mapsto x, Y \mapsto y] = [X \mapsto 0, Y \mapsto !x \cdot y]
\]

and show that

\[
w \in S \implies f(w) \in S.
\]
Topic 5

PCF
Types

\[ \tau ::= \text{nat} \mid \text{bool} \mid \tau \rightarrow \tau \]
PCF syntax

Types

\[ \tau ::= \text{nat} \mid \text{bool} \mid \tau \rightarrow \tau \]

Expressions

\[ M ::= 0 \mid \text{succ}(M) \mid \text{pred}(M) \]
PCF syntax

Types

\[ \tau ::= \text{nat} \mid \text{bool} \mid \tau \to \tau \]

Expressions

\[ M ::= 0 \mid \text{succ}(M) \mid \text{pred}(M) \]
\[ \mid \text{true} \mid \text{false} \mid \text{zero}(M) \]
PCF syntax

Types

\[ \tau ::= \text{nat} \mid \text{bool} \mid \tau \rightarrow \tau \]

Expressions

\[ M ::= 0 \mid \text{succ}(M) \mid \text{pred}(M) \]

\[ \mid \text{true} \mid \text{false} \mid \text{zero}(M) \]

\[ \mid x \mid \text{if } M \text{ then } M \text{ else } M \]
PCF syntax

Types

\[ \tau ::= \text{nat} \mid \text{bool} \mid \tau \rightarrow \tau \]

Expressions

\[ M ::= 0 \mid \text{succ}(M) \mid \text{pred}(M) \mid \text{true} \mid \text{false} \mid \text{zero}(M) \mid x \mid \text{if } M \text{ then } M \text{ else } M \mid \text{fn } x : \tau . M \mid M M \mid \text{fix}(M) \]

where \( x \in \mathbb{V} \), an infinite set of variables.
PCF syntax

Types

\[ \tau ::= \text{nat} \mid \text{bool} \mid \tau \to \tau \]

Expressions

\[ M ::= 0 \mid \text{succ}(M) \mid \text{pred}(M) \]
\[ \mid \text{true} \mid \text{false} \mid \text{zero}(M) \]
\[ \mid x \mid \text{if } M \text{ then } M \text{ else } M \]
\[ \mid \text{fn } x : \tau . M \mid M \ M \mid \text{fix}(M) \]

where \( x \in \mathbb{V} \), an infinite set of variables.

**Technicality:** We identify expressions up to \( \alpha \)-conversion of bound variables (created by the \text{fn} expression-former): by definition a PCF term is an \( \alpha \)-equivalence class of expressions.
PCF typing relation, $\Gamma \vdash M : \tau$

- $\Gamma$ is a type environment, i.e. a finite partial function mapping variables to types (whose domain of definition is denoted $\text{dom}(\Gamma)$)
- $M$ is a term
- $\tau$ is a type.
PCF typing relation, \( \Gamma \vdash M : \tau \)

- \( \Gamma \) is a **type environment**, i.e. a finite partial function mapping variables to types (whose domain of definition is denoted \( \text{dom}(\Gamma) \))

- \( M \) is a term

- \( \tau \) is a **type**.

**Notation:**

\( M : \tau \) means \( M \) is closed and \( \emptyset \vdash M : \tau \) holds.

\[ \text{PCF}_\tau \overset{\text{def}}{=} \{ M \mid M : \tau \} \]
PCF typing relation (sample rules)

\[
\begin{align*}
    & (\text{fn}) \\
    & \frac{\Gamma[x \mapsto \tau] \vdash M : \tau'}{
    \Gamma \vdash \text{fn} \ x : \tau . \ M : \tau \rightarrow \tau'} \\
    & \text{if } x \notin \text{dom}(\Gamma)
\end{align*}
\]
PCF typing relation (sample rules)

\[ (\text{fn}) \quad \frac{\Gamma[x \mapsto \tau] \vdash M : \tau'}{
\Gamma \vdash \text{fn} \, x : \tau \cdot M : \tau \rightarrow \tau'} \quad \text{if } x \notin \text{dom}(\Gamma) \]

\[ (\text{app}) \quad \frac{\Gamma \vdash M_1 : \tau \rightarrow \tau' \quad \Gamma \vdash M_2 : \tau}{\Gamma \vdash M_1 \, M_2 : \tau'} \]
PCF typing relation (sample rules)

\[
\begin{align*}
(\text{:\text{fn}}) & \quad \frac{\Gamma[x \mapsto \tau] \vdash M : \tau'}{\Gamma \vdash \text{fn} \, x : \tau \cdot M : \tau \rightarrow \tau'} \quad \text{if } x \notin \text{dom}(\Gamma) \\
(\text{:\text{app}}) & \quad \frac{\Gamma \vdash M_1 : \tau \rightarrow \tau' \quad \Gamma \vdash M_2 : \tau}{\Gamma \vdash M_1 \, M_2 : \tau'} \\
(\text{:\text{fix}}) & \quad \frac{\Gamma \vdash M : \tau \rightarrow \tau}{\Gamma \vdash \text{fix}(M) : \tau}
\end{align*}
\]
Partial recursive functions in PCF

- Primitive recursion.

\[
\begin{align*}
h(x, 0) &= f(x) \\
h(x, y + 1) &= g(x, y, h(x, y))
\end{align*}
\]
Partial recursive functions in PCF

- Primitive recursion.

\[
\begin{align*}
  h(x, 0) &= f(x) \\
  h(x, y + 1) &= g(x, y, h(x, y))
\end{align*}
\]

- Minimisation.

\[ m(x) = \text{the least } y \geq 0 \text{ such that } k(x, y) = 0 \]
PCF evaluation relation

takes the form

\[ M \downarrow^\tau V \]

where

- \( \tau \) is a PCF type
- \( M, V \in \text{PCF}_\tau \) are closed PCF terms of type \( \tau \)
- \( V \) is a value,

\[ V ::= 0 \mid \text{succ}(V) \mid \text{true} \mid \text{false} \mid \text{fn } x : \tau . M. \]
PCF evaluation (sample rules)

\[ (\downarrow_{\text{val}}) \quad V \downarrow_{\tau} V \quad (V \text{ a value of type } \tau) \]
PCF evaluation (sample rules)

\[(\Downarrow_{\text{val}}) \quad V \Downarrow_\tau V \quad (V \text{ a value of type } \tau)\]

\[
(\Downarrow_{\text{cbn}}) \quad \frac{M_1 \Downarrow_{\tau \rightarrow \tau'} \text{ fn } x : \tau . M'_1 \quad M'_1[M_2/x] \Downarrow_{\tau'} V}{M_1 M_2 \Downarrow_{\tau'} V}
\]
PCF evaluation (sample rules)

\[ (\downarrow_{\text{val}}) \quad V \downarrow_\tau V \quad (V \text{ a value of type } \tau) \]

\[ (\downarrow_{\text{cbn}}) \quad M_1 \downarrow_\tau \text{fix} \quad \text{fn } x : \tau \cdot M'_1 \quad M'_1[M_2/x] \downarrow_\tau' V \]

\[ M_1 M_2 \downarrow_\tau' V \]

\[ (\downarrow_{\text{fix}}) \quad M \text{fix}(M) \downarrow_\tau V \]

\[ \text{fix}(M) \downarrow_\tau V \]
Two phrases of a programming language are contextually equivalent if any occurrences of the first phrase in a complete program can be replaced by the second phrase without affecting the observable results of executing the program.
Contextual equivalence of PCF terms

Given PCF terms $M_1, M_2$, PCF type $\tau$, and a type environment $\Gamma$, the relation $\Gamma \vdash M_1 \equiv_{ctx} M_2 : \tau$ is defined to hold iff

- Both the typings $\Gamma \vdash M_1 : \tau$ and $\Gamma \vdash M_2 : \tau$ hold.

- For all PCF contexts $C$ for which $C[M_1]$ and $C[M_2]$ are closed terms of type $\gamma$, where $\gamma = \text{nat}$ or $\gamma = \text{bool}$, and for all values $V : \gamma$,

$$C[M_1] \Downarrow_\gamma V \iff C[M_2] \Downarrow_\gamma V.$$
PCF denotational semantics — aims
PCF denotational semantics — aims

- PCF types $\tau \mapsto$ domains $[\tau]$. 
PCF denotational semantics — aims

- PCF types $\tau \mapsto$ domains $[[\tau]]$.

- Closed PCF terms $M : \tau \mapsto$ elements $[[M]] \in [[\tau]]$.

  Denotations of open terms will be continuous functions.
PCF denotational semantics — aims

- PCF types $\tau \mapsto$ domains $[\tau]$.

- Closed PCF terms $M : \tau \mapsto$ elements $[M] \in [\tau]$.
  Denotations of open terms will be continuous functions.

- Compositionality.
  In particular: $[M] = [M'] \Rightarrow [C[M]] = [C[M']]$. 
PCF denotational semantics — aims

- PCF types $\tau \mapsto$ domains $[[\tau]]$.

- Closed PCF terms $\mathcal{M} : \tau \mapsto$ elements $[[\mathcal{M}]] \in [[\tau]]$.
  Denotations of open terms will be continuous functions.

- Compositionality.
  In particular: $[[\mathcal{M}]] = [[\mathcal{M}']] \Rightarrow [[\mathcal{C}[\mathcal{M}]]] = [[\mathcal{C}[\mathcal{M}']]].$

- Soundness.
  For any type $\tau$, $\mathcal{M} \Downarrow_{\tau} V \Rightarrow [[\mathcal{M}]] = [[V]].$
PCF denotational semantics — aims

- PCF types \( \tau \mapsto \text{domains } [\tau] \).

- Closed PCF terms \( M : \tau \mapsto \text{elements } [M] \in [\tau] \).
  Denotations of open terms will be continuous functions.

- Compositionality.
  In particular: \([M] = [M'] \Rightarrow [C[M]] = [C[M']]\).

- Soundness.
  For any type \( \tau \), \( M \Downarrow_\tau V \Rightarrow [M] = [V] \).

- Adequacy.
  For \( \tau = \text{bool or nat} \), \([M] = [V] \in [\tau] \implies M \Downarrow_\tau V \).
Theorem. For all types $\tau$ and closed terms $M_1, M_2 \in \text{PCF}_\tau$, if $\llbracket M_1 \rrbracket$ and $\llbracket M_2 \rrbracket$ are equal elements of the domain $\llbracket \tau \rrbracket$, then $M_1 \simeq_{\text{ctx}} M_2 : \tau$. 
**Theorem.** For all types $\tau$ and closed terms $M_1, M_2 \in \text{PCF}_\tau$, if $[M_1]$ and $[M_2]$ are equal elements of the domain $[\tau]$, then $M_1 \simeq_{\text{ctx}} M_2 : \tau$.

**Proof.**

$$C[M_1] \downarrow_{\text{nat}} V \Rightarrow [C[M_1]] = [V] \quad \text{(soundness)}$$

$$\Rightarrow [C[M_2]] = [V] \quad \text{(compositionality on } [M_1] = [M_2] \text{)}$$

$$\Rightarrow C[M_2] \downarrow_{\text{nat}} V \quad \text{(adequacy)}$$

and symmetrically. □
Proof principle

To prove

\[ M_1 \simeq_{\text{ctx}} M_2 : \tau \]

it suffices to establish

\[ \llbracket M_1 \rrbracket = \llbracket M_2 \rrbracket \text{ in } \llbracket \tau \rrbracket \]
Proof principle

To prove

\[ M_1 \sim_{ctx} M_2 : \tau \]

it suffices to establish

\[ \llbracket M_1 \rrbracket = \llbracket M_2 \rrbracket \text{ in } \llbracket \tau \rrbracket \]

? The proof principle is sound, but is it complete? That is, is equality in the denotational model also a necessary condition for contextual equivalence?
Topic 6

Denotational Semantics of PCF
Denotational semantics of PCF

To every typing judgement

$$\Gamma \vdash M : \tau$$

we associate a continuous function

$$[\Gamma \vdash M] : [[\Gamma]] \to [[\tau]]$$

between domains.
Denotational semantics of PCF types

\[ [\text{nat}] \overset{\text{def}}{=} \mathbb{N}_\perp \]  
   (flat domain)

\[ [\text{bool}] \overset{\text{def}}{=} \mathbb{B}_\perp \]  
   (flat domain)

where \( \mathbb{N} = \{0, 1, 2, \ldots\} \) and \( \mathbb{B} = \{\text{true, false}\} \).
Denotational semantics of PCF types

\[[nat] \overset{\text{def}}{=} \mathbb{N}_\perp\] (flat domain)

\[[bool] \overset{\text{def}}{=} \mathbb{B}_\perp\] (flat domain)

\[[\tau \rightarrow \tau'] \overset{\text{def}}{=} [\tau] \rightarrow [\tau']\] (function domain).

where $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $\mathbb{B} = \{true, false\}$. 
Denotational semantics of PCF type environments

\[
\llbracket \Gamma \rrbracket \overset{\text{def}}{=} \prod_{x \in \text{dom}(\Gamma)} \llbracket \Gamma(x) \rrbracket \quad (\Gamma\text{-environments})
\]
Denotational semantics of PCF type environments

\[ [\Gamma] \overset{\text{def}}{=} \prod_{x \in \text{dom}(\Gamma)} [\Gamma(x)] \quad (\Gamma\text{-environments}) \]

= the domain of partial functions \( \rho \) from variables to domains such that \( \text{dom}(\rho) = \text{dom}(\Gamma) \) and \( \rho(x) \in [\Gamma(x)] \) for all \( x \in \text{dom}(\Gamma) \)
Denotational semantics of PCF type environments

\[ [\Gamma] \overset{\text{def}}{=} \prod_{x \in \text{dom}(\Gamma)} [\Gamma(x)] \quad (\Gamma\text{-environments}) \]

\[ = \text{the domain of partial functions } \rho \text{ from variables to domains such that } \text{dom}(\rho) = \text{dom}(\Gamma) \text{ and } \rho(x) \in [\Gamma(x)] \text{ for all } x \in \text{dom}(\Gamma) \]

Example:

1. For the empty type environment \( \emptyset \),

\[ [[\emptyset]] = \{ \perp \} \]

where \( \perp \) denotes the unique partial function with \( \text{dom}(\perp) = \emptyset \).
2. $\llbracket \langle x \mapsto \tau \rangle \rrbracket = \left( \{ x \} \rightarrow \llbracket \tau \rrbracket \right)$
2. $[\{x \mapsto \tau\}] = (\{x\} \rightarrow [\tau]) \cong [\tau]$
2. $\llbracket \langle x \mapsto \tau \rangle \rrbracket = \left( \{ x \} \to [\tau] \right) \cong [\tau]$

3. 

$\llbracket \langle x_1 \mapsto \tau_1, \ldots, x_n \mapsto \tau_n \rangle \rrbracket$

$\cong \left( \{ x_1 \} \to [\tau_1] \right) \times \ldots \times \left( \{ x_n \} \to [\tau_n] \right)$

$\cong [\tau_1] \times \ldots \times [\tau_n]$
Denotational semantics of PCF terms, I

\[
[\Gamma \vdash 0](\rho) \overset{\text{def}}{=} 0 \in [\text{nat}]
\]

\[
[\Gamma \vdash \text{true}](\rho) \overset{\text{def}}{=} \text{true} \in [\text{bool}]
\]

\[
[\Gamma \vdash \text{false}](\rho) \overset{\text{def}}{=} \text{false} \in [\text{bool}]
\]
Denotational semantics of PCF terms, I

\[ [\Gamma \vdash 0](\rho) \overset{\text{def}}{=} 0 \in \mathbb{nat} \]

\[ [\Gamma \vdash \text{true}](\rho) \overset{\text{def}}{=} \text{true} \in \mathbb{bool} \]

\[ [\Gamma \vdash \text{false}](\rho) \overset{\text{def}}{=} \text{false} \in \mathbb{bool} \]

\[ [\Gamma \vdash x](\rho) \overset{\text{def}}{=} \rho(x) \in [\Gamma(x)] \quad (x \in \text{dom}(\Gamma)) \]
\[
\llbracket \Gamma \vdash \text{succ}(M) \rrbracket(\rho)
\]

\[
\text{def} \quad \begin{cases} 
\llbracket \Gamma \vdash M \rrbracket(\rho) + 1 & \text{if } \llbracket \Gamma \vdash M \rrbracket(\rho) \neq \bot \\
\bot & \text{if } \llbracket \Gamma \vdash M \rrbracket(\rho) = \bot
\end{cases}
\]
Denotational semantics of PCF terms, II

\[ \llbracket \Gamma \vdash \text{succ}(M) \rrbracket(\rho) \]

\[ \overset{\text{def}}{=} \begin{cases} \llbracket \Gamma \vdash M \rrbracket(\rho) + 1 & \text{if } \llbracket \Gamma \vdash M \rrbracket(\rho) \neq \bot \\ \bot & \text{if } \llbracket \Gamma \vdash M \rrbracket(\rho) = \bot \end{cases} \]

\[ \llbracket \Gamma \vdash \text{pred}(M) \rrbracket(\rho) \]

\[ \overset{\text{def}}{=} \begin{cases} \llbracket \Gamma \vdash M \rrbracket(\rho) - 1 & \text{if } \llbracket \Gamma \vdash M \rrbracket(\rho) > 0 \\ \bot & \text{if } \llbracket \Gamma \vdash M \rrbracket(\rho) = 0, \bot \end{cases} \]
Denotational semantics of PCF terms, II

\[
[\Gamma \vdash \text{succ}(M)](\rho) \quad \text{def} \quad \begin{cases} 
[\Gamma \vdash M](\rho) + 1 & \text{if } [\Gamma \vdash M](\rho) \neq \bot \\
\bot & \text{if } [\Gamma \vdash M](\rho) = \bot
\end{cases}
\]

\[
[\Gamma \vdash \text{pred}(M)](\rho) \quad \text{def} \quad \begin{cases} 
[\Gamma \vdash M](\rho) - 1 & \text{if } [\Gamma \vdash M](\rho) > 0 \\
\bot & \text{if } [\Gamma \vdash M](\rho) = 0, \bot
\end{cases}
\]

\[
[\Gamma \vdash \text{zero}(M)](\rho) \quad \text{def} \quad \begin{cases} 
\text{true} & \text{if } [\Gamma \vdash M](\rho) = 0 \\
\text{false} & \text{if } [\Gamma \vdash M](\rho) > 0 \\
\bot & \text{if } [\Gamma \vdash M](\rho) = \bot
\end{cases}
\]
Denotational semantics of PCF terms, III

\[ [\Gamma \vdash \textbf{if } M_1 \textbf{ then } M_2 \textbf{ else } M_3](\rho) \]

def\[ = \begin{cases} 
[\Gamma \vdash M_2](\rho) & \text{if } [\Gamma \vdash M_1](\rho) = \text{true} \\
[\Gamma \vdash M_3](\rho) & \text{if } [\Gamma \vdash M_1](\rho) = \text{false} \\
\bot & \text{if } [\Gamma \vdash M_1](\rho) = \bot
\end{cases} \]
Denotational semantics of PCF terms, III

\[
\llbracket \Gamma \vdash \textbf{if } M_1 \textbf{ then } M_2 \textbf{ else } M_3 \rrbracket (\rho) \]

\[
\overset{\text{def}}{=} \begin{cases} 
\llbracket \Gamma \vdash M_2 \rrbracket (\rho) & \text{if } \llbracket \Gamma \vdash M_1 \rrbracket (\rho) = \text{true} \\
\llbracket \Gamma \vdash M_3 \rrbracket (\rho) & \text{if } \llbracket \Gamma \vdash M_1 \rrbracket (\rho) = \text{false} \\
\bot & \text{if } \llbracket \Gamma \vdash M_1 \rrbracket (\rho) = \bot 
\end{cases}
\]

\[
\llbracket \Gamma \vdash M_1 M_2 \rrbracket (\rho) \overset{\text{def}}{=} \left( \llbracket \Gamma \vdash M_1 \rrbracket (\rho) \right) \left( \llbracket \Gamma \vdash M_2 \rrbracket (\rho) \right)
\]
Denotational semantics of PCF terms, IV

\[
\left[ \Gamma \vdash \text{fn } x : \tau . M \right](\rho) \\
\overset{\text{def}}{=} \lambda d \in \left[ \tau \right]. \left[ \Gamma[x \mapsto \tau] \vdash M \right](\rho[x \mapsto d]) \tag{\(x \notin \text{dom}(\Gamma)\)}
\]

\textbf{NB:} \(\rho[x \mapsto d] \in \left[ \Gamma[x \mapsto \tau] \right]\) is the function mapping \(x\) to \(d \in \left[ \tau \right]\) and otherwise acting like \(\rho\).
Recall that \( \text{fix} \) is the function assigning least fixed points to continuous functions.
Denotational semantics of PCF

**Proposition.** For all typing judgements $\Gamma \vdash M : \tau$, the denotation

$$[\Gamma \vdash M] : [\Gamma] \to [\tau]$$

is a well-defined continuous function.
Denotations of closed terms

For a closed term $M \in \text{PCF}_\tau$, we get

$$[[\emptyset \vdash M]] : [[\emptyset]] \rightarrow [[\tau]]$$

and, since $[[\emptyset]] = \{\bot\}$, we have

$$[[M]] \overset{\text{def}}{=} [[\emptyset \vdash M]](\bot) \in [[\tau]] \quad (M \in \text{PCF}_\tau)$$
Compositionality

**Proposition.** For all typing judgements $\Gamma \vdash M : \tau$ and $\Gamma \vdash M' : \tau$, and all contexts $C[-]$ such that $\Gamma' \vdash C[M] : \tau'$ and $\Gamma' \vdash C[M'] : \tau'$,

if $\llbracket \Gamma \vdash M \rrbracket = \llbracket \Gamma \vdash M' \rrbracket : \llbracket \Gamma \rrbracket \rightarrow \llbracket \tau \rrbracket$

then $\llbracket \Gamma' \vdash C[M] \rrbracket = \llbracket \Gamma' \vdash C[M] \rrbracket : \llbracket \Gamma' \rrbracket \rightarrow \llbracket \tau' \rrbracket$
Soundness

**Proposition.** For all closed terms $M, V \in \text{PCF}_\tau$, if $M \Downarrow_\tau V$ then $[M] = [V] \in [\tau]$. 
Proposition. Suppose that $\Gamma \vdash M : \tau$ and that
$\Gamma[x \mapsto \tau] \vdash M' : \tau'$, so that we also have $\Gamma \vdash M'[M/x] : \tau'$.

Then,

$$
\left[\left[\Gamma \vdash M'[M/x]\right]\right](\rho)
= \left[\left[\Gamma[x \mapsto \tau] \vdash M'\right]\right](\rho[x \mapsto \left[\left[\Gamma \vdash M\right]\right]])
$$

for all $\rho \in \left[\left[\Gamma\right]\right]$. 

Substitution property

Proposition. Suppose that $\Gamma \vdash M : \tau$ and that
$\Gamma[x \mapsto \tau] \vdash M' : \tau'$, so that we also have $\Gamma \vdash M'[M/x] : \tau'$.

Then,

$$
\left[\left[\Gamma \vdash M'[M/x]\right]\right](\rho)
= \left[\left[\Gamma[x \mapsto \tau] \vdash M'\right]\right](\rho[x \mapsto \left[\left[\Gamma \vdash M\right]\right]])
$$

for all $\rho \in \left[\left[\Gamma\right]\right]$. 

Substitution property
Substitution property

**Proposition.** Suppose that $\Gamma \vdash M : \tau$ and that $\Gamma[x \mapsto \tau] \vdash M' : \tau'$, so that we also have $\Gamma \vdash M'[M/x] : \tau'$. Then,

$$\left[ \left[ \Gamma \vdash M'[M/x] \right] \right](\rho) = \left[ \left[ \Gamma[x \mapsto \tau] \vdash M' \right] \right](\rho[x \mapsto [\Gamma \vdash M]])$$

for all $\rho \in [\Gamma]$.

In particular when $\Gamma = \emptyset$, $\left[ \langle x \mapsto \tau \rangle \vdash M' \right] : [\tau] \rightarrow [\tau']$ and

$$\left[ M'[M/x] \right] = \left[ \langle x \mapsto \tau \rangle \vdash M' \right](\left[ M \right])$$
Topic 7

Relating Denotational and Operational Semantics
Adequacy

For any closed PCF terms $M$ and $V$ of *ground* type $\gamma \in \{nat, bool\}$ with $V$ a value

\[ [M] = [V] \in [\gamma] \implies M \Downarrow_{\gamma} V. \]
Adequacy

For any closed PCF terms $M$ and $V$ of ground type $\gamma \in \{\text{nat}, \text{bool}\}$ with $V$ a value

$$\llbracket M \rrbracket = \llbracket V \rrbracket \in \llbracket \gamma \rrbracket \implies M \downarrow^\gamma V.$$ 

**NB.** Adequacy does not hold at function types
Adequacy

For any closed PCF terms $M$ and $V$ of *ground* type $\gamma \in \{nat, bool\}$ with $V$ a value

$$[M] = [V] \in [\gamma] \implies M \downarrow_{\gamma} V.$$ 

**NB.** Adequacy does not hold at function types:

$$[\texttt{fn } x : \tau. (\texttt{fn } y : \tau. y) x] = [\texttt{fn } x : \tau. x] : [\tau] \rightarrow [\tau]$$
For any closed PCF terms $M$ and $V$ of ground type $\gamma \in \{\text{nat}, \text{bool}\}$ with $V$ a value

$$[M] = [V] \in [\gamma] \implies M \downarrow_\gamma V.$$ 

**NB.** Adequacy does not hold at function types:

$$[\text{fn } x : \tau. (\text{fn } y : \tau. y) x] = [\text{fn } x : \tau. x] : [\tau] \rightarrow [\tau]$$

but

$$\text{fn } x : \tau. (\text{fn } y : \tau. y) x \not\downarrow_{\tau \rightarrow \tau} \text{fn } x : \tau. x$$
Adequacy proof idea
1. We cannot proceed to prove the adequacy statement by a straightforward induction on the structure of terms.

Consider $M$ to be $M_1 M_2$, $\text{fix}(M')$. 
Adequacy proof idea

1. We cannot proceed to prove the adequacy statement by a straightforward induction on the structure of terms.

   ▶ Consider $M$ to be $M_1 \ M_2$, $\text{fix}(M')$.

2. So we proceed to prove a stronger statement that applies to terms of arbitrary types and implies adequacy.
Adequacy proof idea

1. We cannot proceed to prove the adequacy statement by a straightforward induction on the structure of terms.
   - Consider $M$ to be $M_1 M_2$, $\text{fix}(M')$.

2. So we proceed to prove a stronger statement that applies to terms of arbitrary types and implies adequacy.
   This statement roughly takes the form:
   \[
   \left[ [M] \right] \prec_\tau M \text{ for all types } \tau \text{ and all } M \in \text{PCF}_\tau
   \]
   where the formal approximation relations
   \[
   \prec_\tau \subseteq \left[ \tau \right] \times \text{PCF}_\tau
   \]
   are logically chosen to allow a proof by induction.
Requirements on the formal approximation relations, I

We want that, for $\gamma \in \{\text{nat}, \text{bool}\}$,

$$[[M]] \llcorner_{\gamma} M \text{ implies } \forall V ([[M]] = [[V]] \implies M \sqsubseteq_{\gamma} V)$$

adequacy
Definition of $d \triangleleft_\gamma M$ ($d \in [\gamma], M \in \text{PCF}_\gamma$) for $\gamma \in \{\text{nat}, \text{bool}\}$

$n \triangleleft_{\text{nat}} M \overset{\text{def}}{\iff} (n \in \mathbb{N} \Rightarrow M \downarrow_{\text{nat}} \text{succ}^n(0))$

$b \triangleleft_{\text{bool}} M \overset{\text{def}}{\iff} (b = \text{true} \Rightarrow M \downarrow_{\text{bool}} \text{true})$

& (b = \text{false} \Rightarrow M \downarrow_{\text{bool}} \text{false})
Proof of: $[M] \triangleleft_{\gamma} M$ implies adequacy

Case $\gamma = \text{nat}$.

$[M] = [V]$

$\implies [M] = [\text{succ}^n(0)]$ for some $n \in \mathbb{N}$

$\implies n = [M] \triangleleft_{\gamma} M$

$\implies M \downarrow \text{succ}^n(0)$ by definition of $\triangleleft_{\text{nat}}$

Case $\gamma = \text{bool}$ is similar.
Requirements on the formal approximation relations, II

We want to be able to proceed by induction.

Consider the case $M = M_1 M_2$. 

$\sim$ \textit{logical definition}
Definition of

\[ f \triangleleft_{\tau \rightarrow \tau'} M \ (f \in ([\tau] \rightarrow [\tau']), M \in PCF_{\tau \rightarrow \tau'}) \]
Definition of

\[ f \triangleleft_{\tau \rightarrow \tau'} M \quad (f \in ([\tau] \rightarrow [\tau']), \ M \in \text{PCF}_{\tau \rightarrow \tau'}) \]

\[ f \triangleleft_{\tau \rightarrow \tau'} M \]

\[ \overset{\text{def}}{\iff} \quad \forall x \in [\tau], N \in \text{PCF}_{\tau} \]

\[ (x \triangleleft_{\tau} N \implies f(x) \triangleleft_{\tau'} M N) \]
Requirements on the formal approximation relations, III

We want to be able to proceed by induction.

Consider the case $M = \text{fix}(M')$.

$\sim$ admissibility property
Admissibility property

**Lemma.** For all types $\tau$ and $M \in \text{PCF}_\tau$, the set

$$\{ d \in [\tau] \mid d \triangleleft_\tau M \}$$

is an admissible subset of $[\tau]$. 
Further properties

Lemma. For all types $\tau$, elements $d, d' \in \llbracket \tau \rrbracket$, and terms $M, N, V \in \text{PCF}_\tau$,

1. If $d \sqsubseteq d'$ and $d' \triangleleft_\tau M$ then $d \triangleleft_\tau M$.

2. If $d \triangleleft_\tau M$ and $\forall V (M \Downarrow_\tau V \implies N \Downarrow_\tau V)$ then $d \triangleleft_\tau N$. 
Requirements on the formal approximation relations, IV

We want to be able to proceed by induction.

Consider the case $M = \text{fn } x : \tau . M'$.

$\rightsquigarrow$ substitutivity property for open terms
Fundamental property

Theorem. For all $\Gamma = \langle x_1 \mapsto \tau_1, \ldots, x_n \mapsto \tau_n \rangle$ and all $\Gamma \vdash M : \tau$, if $d_1 \triangleleft_{\tau_1} M_1, \ldots, d_n \triangleleft_{\tau_n} M_n$ then $[\Gamma \vdash M][x_1 \mapsto d_1, \ldots, x_n \mapsto d_n] \triangleleft_{\tau} M[M_1/x_1, \ldots, M_n/x_n]$.
**Theorem.** For all \( \Gamma = \langle x_1 \mapsto \tau_1, \ldots, x_n \mapsto \tau_n \rangle \) and all \( \Gamma \vdash M : \tau \), if \( d_1 \triangleleft \tau_1 M_1, \ldots, d_n \triangleleft \tau_n M_n \) then
\[ [\Gamma \vdash M][x_1 \mapsto d_1, \ldots, x_n \mapsto d_n] \triangleleft \tau M[M_1/x_1, \ldots, M_n/x_n]. \]

**NB.** The case \( \Gamma = \emptyset \) reduces to
\[ [M] \triangleleft \tau M \]
for all \( M \in \text{PCF}_\tau \).
Fundamental property of the relations $\triangleleft_\tau$

**Proposition.** If $\Gamma \vdash M : \tau$ is a valid PCF typing, then for all $\Gamma$-environments $\rho$ and all $\Gamma$-substitutions $\sigma$

$$\rho \triangleleft_\Gamma \sigma \Rightarrow [\Gamma \vdash M](\rho) \triangleleft_\tau M[\sigma]$$

- $\rho \triangleleft_\Gamma \sigma$ means that $\rho(x) \triangleleft_{\Gamma(x)} \sigma(x)$ holds for each $x \in \text{dom}(\Gamma)$.
- $M[\sigma]$ is the PCF term resulting from the simultaneous substitution of $\sigma(x)$ for $x$ in $M$, each $x \in \text{dom}(\Gamma)$. 
Contextual preorder between PCF terms

Given PCF terms $M_1, M_2$, PCF type $\tau$, and a type environment $\Gamma$, the relation $\Gamma \vdash M_1 \leq_{\text{ctx}} M_2 : \tau$ is defined to hold iff

- Both the typings $\Gamma \vdash M_1 : \tau$ and $\Gamma \vdash M_2 : \tau$ hold.

- For all PCF contexts $C$ for which $C[M_1]$ and $C[M_2]$ are closed terms of type $\gamma$, where $\gamma = \text{nat}$ or $\gamma = \text{bool}$, and for all values $V \in \text{PCF}_\gamma$,

  $$C[M_1] \downarrow_\gamma V \implies C[M_2] \downarrow_\gamma V.$$
Extensionality properties of $\leq_{\text{ctx}}$

At a ground type $\gamma \in \{\text{bool}, \text{nat}\}$,

\[ M_1 \leq_{\text{ctx}} M_2 : \gamma \text{ holds if and only if } \forall V \in \text{PCF}_\gamma (M_1 \downarrow_\gamma V \implies M_2 \downarrow_\gamma V) . \]

At a function type $\tau \rightarrow \tau'$,

\[ M_1 \leq_{\text{ctx}} M_2 : \tau \rightarrow \tau' \text{ holds if and only if } \forall M \in \text{PCF}_\tau (M_1 M \leq_{\text{ctx}} M_2 M : \tau') . \]
Topic 8

Full Abstraction
Proof principle

For all types $\tau$ and closed terms $M_1, M_2 \in \text{PCF}_{\tau}$,

$$[[M_1]] = [[M_2]] \text{ in } [[\tau]] \implies M_1 \simeq_{\text{ctx}} M_2 : \tau .$$

Hence, to prove

$$M_1 \simeq_{\text{ctx}} M_2 : \tau$$

it suffices to establish

$$[[M_1]] = [[M_2]] \text{ in } [[\tau]] .$$
A denotational model is said to be *fully abstract* whenever denotational equality characterises contextual equivalence.
A denotational model is said to be *fully abstract* whenever denotational equality characterises contextual equivalence.

The domain model of **PCF** is *not* fully abstract.

In other words, there are contextually equivalent **PCF** terms with different denotations.
Failure of full abstraction, idea

We will construct two closed terms

$$T_1, T_2 \in \text{PCF}(\text{bool} \rightarrow (\text{bool} \rightarrow \text{bool})) \rightarrow \text{bool}$$

such that

$$T_1 \simeq_{\text{ctx}} T_2$$

and

$$[T_1] \neq [T_2]$$
We achieve $T_1 \cong_{ctx} T_2$ by making sure that

$$\forall M \in \text{PCF}_{\text{bool} \to (\text{bool} \to \text{bool})} \left( T_1 \ M \not\Downarrow_{\text{bool}} \land T_2 \ M \not\Downarrow_{\text{bool}} \right)$$
We achieve $T_1 \approx_{ctx} T_2$ by making sure that

$$\forall M \in \text{PCF}_{\text{bool} \rightarrow (\text{bool} \rightarrow \text{bool})} \left( T_1 M \uparrow_{\text{bool}} \& T_2 M \uparrow_{\text{bool}} \right)$$

Hence,

$$[[T_1]]([[M]]) = \bot = [[T_2]]([[M]])$$

for all $M \in \text{PCF}_{\text{bool} \rightarrow (\text{bool} \rightarrow \text{bool})}$. 
We achieve $T_1 \simeq_{\text{ctx}} T_2$ by making sure that

$$
\forall M \in \text{PCF}_{\text{bool} \to (\text{bool} \to \text{bool})} \left( \langle T_1 \ M \rangle_{\text{bool}} \not\ll \text{bool} \ \& \ \langle T_2 \ M \rangle_{\text{bool}} \not\ll \text{bool} \right)
$$

Hence,

$$
\llbracket T_1 \rrbracket(\llbracket M \rrbracket) = \perp = \llbracket T_2 \rrbracket(\llbracket M \rrbracket)
$$

for all $M \in \text{PCF}_{\text{bool} \to (\text{bool} \to \text{bool})}$.

We achieve $\llbracket T_1 \rrbracket \neq \llbracket T_2 \rrbracket$ by making sure that

$$
\llbracket T_1 \rrbracket(\text{por}) \neq \llbracket T_2 \rrbracket(\text{por})
$$

for some non-definable continuous function

$$
\text{por} \in (\mathbb{B}_{\perp} \to (\mathbb{B}_{\perp} \to \mathbb{B}_{\perp}))
$$
Parallel-or function

is the unique continuous function $por : \mathbb{B}_\perp \to (\mathbb{B}_\perp \to \mathbb{B}_\perp)$ such that

$por \; true \; \perp = true$

$por \; \perp \; true = true$

$por \; false \; false = false$
Parallel-or function

is the unique continuous function $\text{por} : \mathbb{B}_\bot \rightarrow (\mathbb{B}_\bot \rightarrow \mathbb{B}_\bot)$ such that

\[
\begin{align*}
\text{por} \ true \ \bot & = true \\
\text{por} \ \bot \ true & = true \\
\text{por} \ false \ false & = false
\end{align*}
\]

In which case, it necessarily follows by monotonicity that

\[
\begin{align*}
\text{por} \ true \ true & = true & \text{por} \ false \ \bot & = \bot \\
\text{por} \ true \ false & = true & \text{por} \ \bot \ false & = \bot \\
\text{por} \ false \ true & = true & \text{por} \ \bot \ \bot & = \bot
\end{align*}
\]
Proposition. There is no closed PCF term

\[ P : bool \to (bool \to bool) \]

satisfying

\[ [P] = \text{por} : \mathbb{B}_\bot \to (\mathbb{B}_\bot \to \mathbb{B}_\bot) \]
Parallel-or test functions
Parallel-or test functions

For $i = 1, 2$ define

$$T_i \overset{\text{def}}{=} \text{fn } f : \text{bool } \to (\text{bool } \to \text{bool}).$$

\[
\begin{align*}
\quad \text{if } (f \text{ true } \Omega) \text{ then } \\
\quad \quad \text{if } (f \Omega \text{ true}) \text{ then } \\
\quad \quad \quad \text{if } (f \text{ false false}) \text{ then } \Omega \text{ else } B_i \\
\quad \quad \text{else } \Omega \\
\quad \text{else } \Omega
\end{align*}
\]

where $B_1 \overset{\text{def}}{=} \text{true}$, $B_2 \overset{\text{def}}{=} \text{false}$,

and $\Omega \overset{\text{def}}{=} \text{fix(fn } x : \text{bool } . x)$. 
Failure of full abstraction

Proposition.

\[ T_1 \cong_{\text{ctx}} T_2 : (\text{bool} \to (\text{bool} \to \text{bool})) \to \text{bool} \]

\[ [T_1] \neq [T_2] \in (\mathbb{B}_\bot \to (\mathbb{B}_\bot \to \mathbb{B}_\bot)) \to \mathbb{B}_\bot \]
PCF+por

Expressions

\[ M ::= \cdots | \text{por}(M, M) \]

Typing

\[
\begin{align*}
\Gamma &\vdash M_1 : \text{bool} & \Gamma &\vdash M_2 : \text{bool} \\
\Gamma &\vdash \text{por}(M_1, M_2) : \text{bool}
\end{align*}
\]

Evaluation

\[
\begin{align*}
M_1 \Downarrow_{\text{bool}} \text{true} &\quad & M_2 \Downarrow_{\text{bool}} \text{true} \\
\text{por}(M_1, M_2) \Downarrow_{\text{bool}} \text{true} &\quad & \text{por}(M_1, M_2) \Downarrow_{\text{bool}} \text{true} \\
M_1 \Downarrow_{\text{bool}} \text{false} &\quad & M_2 \Downarrow_{\text{bool}} \text{false} \\
\text{por}(M_1, M_2) \Downarrow_{\text{bool}} \text{false}
\end{align*}
\]
Plotkin’s full abstraction result

The denotational semantics of PCF+por is given by extending that of PCF with the clause

$$\llbracket \Gamma \vdash \text{por}(M_1, M_2) \rrbracket(\rho) \overset{\text{def}}{=} \text{por}(\llbracket \Gamma \vdash M_1 \rrbracket(\rho)) (\llbracket \Gamma \vdash M_2 \rrbracket(\rho))$$

This denotational semantics is fully abstract for contextual equivalence of PCF+por terms:

$$\Gamma \vdash M_1 \cong_{\text{ctx}} M_2 : \tau \iff \llbracket \Gamma \vdash M_1 \rrbracket = \llbracket \Gamma \vdash M_2 \rrbracket.$$