Topic 6

Denotational Semantics of PCF

Denotational semantics of PCF

To every typing judgement

$$\Gamma \vdash M : \tau$$

we associate a continuous function

$$[\![\Gamma \vdash M]\!] : [\![\Gamma]\!] \to [\![\tau]\!]$$

between domains.

Denotational semantics of PCF types

$$[\![nat]\!] \stackrel{\text{def}}{=} \mathbb{N}_\perp$$ (flat domain)

$$[\![bool]\!] \stackrel{\text{def}}{=} \mathbb{B}_\perp$$ (flat domain)

$$[\![\tau \to \tau']\!] \stackrel{\text{def}}{=} [\![\tau]\!] \to [\![\tau']\!]$$ (function domain).

where $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $\mathbb{B} = \{true, false\}$.

Denotational semantics of PCF type environments

$$[\![\Gamma]\!] \stackrel{\text{def}}{=} \prod_{x \in dom(\Gamma)} [\![\Gamma(x)]\!] \qquad (\Gamma\text{-environments})$$

$\rho(x_i) \in [\![\tau_i]\!]$ $\qquad$ $\Gamma(x_i) = \tau_i$

$$\Gamma \left\langle \frac{\{x_i \mapsto \tau_i\}_{i=1,\ldots,n} \qquad dom(\Gamma) = \{x_1, \ldots, x_n\}}{(x_1 : \tau_1, \ldots, x_n : \tau_n) \qquad x_i \neq x_j \ \forall i \neq j} \right.$$

$$[\![\Gamma]\!] = \prod_{i=1}^{n} [\![\tau_i]\!]$$

$\rho \in [\![\Gamma]\!]$ $\qquad$ $\rho = (\rho_1, \ldots, \rho_n)$ with $\rho_i \in [\![\tau_i]\!]$

Denotational semantics of PCF type environments

$$\begin{aligned}
[\![\Gamma]\!] &\stackrel{\text{def}}{=} \textstyle\prod_{x \in dom(\Gamma)} [\![\Gamma(x)]\!] \qquad \text{(Γ-environments)} \\
&= \text{the domain of partial functions } \rho \text{ from variables} \\
&\ \text{to domains such that } dom(\rho) = dom(\Gamma) \text{ and} \\
&\ \rho(x) \in [\![\Gamma(x)]\!] \text{ for all } x \in dom(\Gamma)
\end{aligned}$$

Example:

1. For the empty type environment $\emptyset$,

$$[\![\emptyset]\!] = \{\, \bot \,\}$$

where $\bot$ denotes the unique partial function with $dom(\bot) = \emptyset$.

2. $[\![\langle x \mapsto \tau \rangle]\!] = (\{ x \} \to [\![\tau]\!]) \cong [\![\tau]\!]$

3.

$$
\begin{aligned}
&[\![\langle x_1 \mapsto \tau_1, \ldots, x_n \mapsto \tau_n \rangle]\!] \\
&\quad \cong (\{ x_1 \} \to [\![\tau_1]\!]) \times \ldots \times (\{ x_n \} \to [\![\tau_n]\!]) \\
&\quad \cong [\![\tau_1]\!] \times \ldots \times [\![\tau_n]\!]
\end{aligned}
$$

$[\![\Gamma \vdash M : \tau]\!] : [\![\Gamma]\!] \longrightarrow [\![\tau]\!]$ cont.

Denotational semantics of PCF terms, I

$[\![\Gamma \vdash M : \tau]\!]\rho \in [\![\tau]\!] \quad \forall \rho \in [\![\Gamma]\!]$

$$[\![\Gamma \vdash \mathbf{0}]\!](\rho) \stackrel{\text{def}}{=} 0 \in [\![nat]\!]$$

$$[\![\Gamma \vdash \mathbf{true}]\!](\rho) \stackrel{\text{def}}{=} true \in [\![bool]\!]$$

$$[\![\Gamma \vdash \mathbf{false}]\!](\rho) \stackrel{\text{def}}{=} false \in [\![bool]\!]$$

Denotational semantics of PCF terms, I

$$[\![\Gamma \vdash \mathbf{0}]\!](\rho) \stackrel{\text{def}}{=} 0 \in [\![nat]\!]$$

$$[\![\Gamma \vdash \mathbf{true}]\!](\rho) \stackrel{\text{def}}{=} true \in [\![bool]\!]$$

$$[\![\Gamma \vdash \mathbf{false}]\!](\rho) \stackrel{\text{def}}{=} false \in [\![bool]\!]$$

$$[\![\Gamma \vdash x]\!](\rho) \stackrel{\text{def}}{=} \rho(x) \in [\![\Gamma(x)]\!] \qquad (x \in dom(\Gamma))$$

$$[\![x_1 : \tau_1, \ldots, x_n : \tau_n \vdash x_i : \tau_i]\!](\rho_1, \ldots, \rho_n) = \rho_i$$

Denotational semantics of PCF terms, II

$$[\![\Gamma \vdash \mathbf{succ}(M)]\!](\rho)$$

$$\stackrel{\text{def}}{=} \begin{cases} [\![\Gamma \vdash M]\!](\rho) + 1 & \text{if } [\![\Gamma \vdash M]\!](\rho) \neq \bot \\ \bot & \text{if } [\![\Gamma \vdash M]\!](\rho) = \bot \end{cases}$$

$$[\![\Gamma \vdash \mathbf{pred}(M)]\!](\rho)$$

$$\stackrel{\text{def}}{=} \begin{cases} [\![\Gamma \vdash M]\!](\rho) - 1 & \text{if } [\![\Gamma \vdash M]\!](\rho) > 0 \\ \bot & \text{if } [\![\Gamma \vdash M]\!](\rho) = 0, \bot \end{cases}$$

$$[\![\Gamma \vdash \mathbf{zero}(M)]\!](\rho) \stackrel{\text{def}}{=} \begin{cases} \mathit{true} & \text{if } [\![\Gamma \vdash M]\!](\rho) = 0 \\ \mathit{false} & \text{if } [\![\Gamma \vdash M]\!](\rho) > 0 \\ \bot & \text{if } [\![\Gamma \vdash M]\!](\rho) = \bot \end{cases}$$

$\forall \rho \in [\![\Gamma]\!]$ $\quad [\![\Gamma \vdash M_1]\!](\rho) \in ([\![\tau_2]\!] \to [\![\tau_1]\!])$

Denotational semantics of PCF terms, III

$[\![\Gamma \vdash M_2]\!](\rho) \in [\![\tau_2]\!]$

$$[\![\Gamma \vdash \textbf{if } M_1 \textbf{ then } M_2 \textbf{ else } M_3]\!](\rho)$$

$$\stackrel{\text{def}}{=} \begin{cases} [\![\Gamma \vdash M_2]\!](\rho) & \text{if } [\![\Gamma \vdash M_1]\!](\rho) = \mathit{true} \\ [\![\Gamma \vdash M_3]\!](\rho) & \text{if } [\![\Gamma \vdash M_1]\!](\rho) = \mathit{false} \\ \perp & \text{if } [\![\Gamma \vdash M_1]\!](\rho) = \perp \end{cases}$$

$$[\![\Gamma \vdash M_1\, M_2]\!](\rho) \stackrel{\text{def}}{=} ([\![\Gamma \vdash M_1]\!](\rho))\, ([\![\Gamma \vdash M_2]\!](\rho))$$

$[\![\Gamma \vdash M_1]\!] : [\![\Gamma]\!] \to ([\![\tau_2]\!] \to [\![\tau_1]\!])$ $\quad [\![\Gamma \vdash M_2]\!] : [\![\Gamma]\!] \to [\![\tau_2]\!]$

$\Gamma \vdash M_1 : \tau_2 \to \tau_1$ $\quad \Gamma \vdash M_2 : \tau_2$

$$\frac{\Gamma[x \mapsto \tau] \vdash M : \sigma}{\Gamma \vdash \mathrm{fn}\, x : \tau .\, M : \tau \to \sigma} \quad x \notin dom(\Gamma)$$

$$[\![\Gamma[x \mapsto \tau] \vdash M]\!] : [\![\Gamma[x \mapsto \tau]]\!] \longrightarrow [\![\sigma]\!]$$

$$\rho' \in [\![\Gamma[x \mapsto \tau]]\!]$$

$$\shortparallel$$

$$\{ x_i \mapsto d_i ,\ x \mapsto d \} \qquad x_i \in dom(\Gamma)$$

$$d_i \in [\![\Gamma(x_i)]\!] \quad d \in [\![\tau]\!]$$

$$[\![\Gamma \vdash \mathrm{fn}\, x . \tau . M]\!](\rho) \in [\![\tau]\!] \to [\![\sigma]\!]$$

$$\shortparallel \ \lambda d \in [\![\tau]\!] .\ [\![\Gamma[x \mapsto \tau] \vdash M]\!] (\rho, [x \mapsto d])$$

Denotational semantics of PCF terms, IV

$$\llbracket \Gamma \vdash \mathbf{fn}\, x : \tau \,.\, M \rrbracket(\rho) \qquad (x \notin dom(\Gamma))$$
$$\stackrel{\text{def}}{=} \lambda d \in \llbracket \tau \rrbracket \,.\, \llbracket \Gamma[x \mapsto \tau] \vdash M \rrbracket(\rho[x \mapsto d])$$

NB: $\rho[x \mapsto d] \in \llbracket \Gamma[x \mapsto \tau] \rrbracket$ is the function mapping x to $d \in \llbracket \tau \rrbracket$ and otherwise acting like ρ.

$$\frac{\Gamma \vdash M : \tau \to \tau}{\Gamma \vdash \mathit{fix}(M) : \tau}$$

$[\![\Gamma \vdash M]\!](\rho) \in ([\![\tau]\!] \to [\![\tau]\!])$

$[\![\Gamma \vdash \mathit{fix}(M)]\!](\rho) \in [\![\tau]\!]$

Denotational semantics of PCF terms, V

$$[\![\Gamma \vdash \mathbf{fix}(M)]\!](\rho) \stackrel{\text{def}}{=} \mathit{fix}([\![\Gamma \vdash M]\!](\rho))$$

Recall that fix is the function assigning least fixed points to continuous functions.

Eg. $\underline{\text{eval}} : (D \to E) \times D \longrightarrow E$

$\underline{\text{eval}}\,(f,d) \stackrel{\text{def}}{=} f(d)$ ~ $\underline{\text{check}}$ eval is continuous.

Denotational semantics of PCF

Proposition. *For all typing judgements* $\Gamma \vdash M : \tau$, *the denotation*

$$[\![\Gamma \vdash M]\!] : [\![\Gamma]\!] \to [\![\tau]\!]$$

is a well-defined continous function.

$(f,d) \sqsubseteq (f',d') \Leftrightarrow f \sqsubseteq f' \wedge d \sqsubseteq d'$

?

$\underline{\text{eval}}\,(f,d) \stackrel{?}{\sqsubseteq} \text{eval}\,(f',d')$

$\quad\; \| \qquad\qquad\qquad \|$

$\quad f(d) \qquad\qquad\; f'(d')$

(f_i, d_i) chain

$\bigsqcup_i f_i \left(\bigsqcup_i d_i\right)$

$= \bigsqcup_i f_i(d_i)$

Denotations of closed terms

For a closed term $M \in \mathrm{PCF}_\tau$, we get

$$[\![\emptyset \vdash M]\!] : [\![\emptyset]\!] \to [\![\tau]\!]$$

and, since $[\![\emptyset]\!] = \{ \perp \}$, we have

$$\boxed{[\![M]\!] \stackrel{\mathrm{def}}{=} [\![\emptyset \vdash M]\!](\perp) \in [\![\tau]\!] \qquad (M \in \mathrm{PCF}_\tau)}$$

$$[\![\Gamma \vdash M]\!] = [\![\Gamma \vdash M']\!]$$

$$\Leftrightarrow \forall \rho \in [\![\Gamma]\!].\ [\![\Gamma \vdash M]\!](\rho) = [\![\Gamma \vdash M']\!](\rho) \text{ in } [\![\tau]\!]$$

Compositionality

Proposition. *For all typing judgements* $\Gamma \vdash M : \tau$ *and* $\Gamma \vdash M' : \tau$, *and all contexts* $\mathcal{C}[-]$ *such that* $\Gamma' \vdash \mathcal{C}[M] : \tau'$ *and* $\Gamma' \vdash \mathcal{C}[M'] : \tau'$,

$$\textit{if} \quad [\![\Gamma \vdash M]\!] = [\![\Gamma \vdash M']\!] : [\![\Gamma]\!] \to [\![\tau]\!]$$

$$\textit{then} \quad [\![\Gamma' \vdash \mathcal{C}[M]]\!] = [\![\Gamma' \vdash \mathcal{C}[M']]\!] : [\![\Gamma']\!] \to [\![\tau']\!]$$

$$\mathcal{C}[-] = \mathrm{succ}(\mathcal{C}'[\ ])$$

$$\frac{M \Downarrow V}{\mathrm{succ}(M) \Downarrow \mathrm{succ}(V)}$$

$$\begin{array}{ccc} [\![\mathrm{succ}(M)]\!] & \stackrel{?}{=} & [\![\mathrm{succ}(V)]\!] \\ \| & & \| \\ \mathrm{succ}\,[\![M]\!] & & \mathrm{succ}\,[\![V]\!] \end{array}$$

by ind. $[\![M]\!] = [\![V]\!]$

Soundness

Proposition. *For all closed terms* $M, V \in \mathrm{PCF}_\tau$,

if $M \Downarrow_\tau V$ *then* $[\![M]\!] = [\![V]\!] \in [\![\tau]\!]$.

$$\frac{M_1 \Downarrow \mathrm{fn}\, x . M \qquad M[{}^{M_2}/_x] \Downarrow V}{M_1\, M_2 \Downarrow V}$$

$$[\![M_1\, M_2]\!] \overset{?}{=} [\![V]\!]$$

$$\parallel$$

$$[\![M_1]\!] \big([\![M_2]\!] \big) \quad ①$$

By ind. $\quad [\![M_1]\!] = [\![\mathrm{fn}\, x . M]\!] \qquad [\![M[{}^{M_2}/_x]]\!] = [\![V]\!]$

$$\parallel$$

$$② \quad \lambda d . [\![M]\!] (x \mapsto d)$$

$\parallel$? Lemma

① & ② $\Rightarrow$

$$[\![M_1]\!] \big([\![M_2]\!] \big) = [\![M]\!] \big(x \mapsto [\![M_2]\!] \big)$$

Substitution property

Proposition. *Suppose that* $\Gamma \vdash M : \tau$ *and that* $\Gamma[x \mapsto \tau] \vdash M' : \tau'$, *so that we also have* $\Gamma \vdash M'[M/x] : \tau'$.

Then,

$$
\begin{aligned}
&[\![\Gamma \vdash M'[M/x]]\!](\rho) \\
&\qquad = [\![\Gamma[x \mapsto \tau] \vdash M']\!]\big(\rho\big[x \mapsto [\![\Gamma \vdash M]\!]\big]\big)
\end{aligned}
$$

for all $\rho \in [\![\Gamma]\!]$.

substitution $\longleftrightarrow$ function application

Substitution property

Proposition. *Suppose that* $\Gamma \vdash M : \tau$ *and that* $\Gamma[x \mapsto \tau] \vdash M' : \tau'$, *so that we also have* $\Gamma \vdash M'[M/x] : \tau'$.

Then,

$$\begin{aligned}&[\![\Gamma \vdash M'[M/x]]\!](\rho) \\ &\quad = [\![\Gamma[x \mapsto \tau] \vdash M']\!]\big(\rho\big[x \mapsto [\![\Gamma \vdash M]\!]\big]\big)\end{aligned}$$

for all $\rho \in [\![\Gamma]\!]$.

In particular when $\Gamma = \emptyset$, $[\![\langle x \mapsto \tau\rangle \vdash M']\!] : [\![\tau]\!] \to [\![\tau']\!]$ and

$$[\![M'[M/x]]\!] = [\![\langle x \mapsto \tau\rangle \vdash M']\!]([\![M]\!])$$

$$\frac{M\,(\mathrm{fix}\,M) \Downarrow V}{\mathrm{fix}(M) \Downarrow V}$$

$$[\![\mathrm{fix}(M)]\!] \stackrel{?}{=} [\![V]\!]$$

Ind. $[\![M\,(\mathrm{fix}\,M)]\!] = [\![V]\!]$

$$\begin{array}{l} \quad\; \| \\ [\![M]\!]\,(\,[\![\underline{\mathrm{fix}}(M)]\!]\,) = [\![M]\!]\,(\mathrm{fix}\,[\![M]\!]) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \| \\ \qquad\qquad\qquad\qquad\qquad\quad \mathrm{fix}\,[\![M]\!] = [\![\mathrm{fix}\,M]\!] \end{array}$$