Topic 5

PCF

PCF syntax

Types

$$\tau ::= nat \mid bool \mid \tau \to \tau$$

Expressions

$$\begin{array}{rcl} M & ::= & \mathbf{0} \mid \mathbf{succ}(M) \mid \mathbf{pred}(M) \\ & \mid & \mathbf{true} \mid \mathbf{false} \mid \mathbf{zero}(M) \\ & \mid & x \mid \mathbf{if}\ M\ \mathbf{then}\ M\ \mathbf{else}\ M \\ & \mid & \mathbf{fn}\, x : \tau \,.\, M \mid M\ M \mid \mathbf{fix}(M) \end{array}$$

where $x \in \mathbb{V}$, an infinite set of variables.

Technicality: We identify expressions up to α-conversion of bound variables (created by the $\mathbf{fn}$ expression-former): by definition a PCF term is an α-equivalence class of expressions.

PCF typing relation (sample rules)

$$(:_{\mathrm{fn}}) \quad \frac{\Gamma[x \mapsto \tau] \vdash M : \tau'}{\Gamma \vdash \mathbf{fn}\, x : \tau \,.\, M : \tau \to \tau'} \quad \text{if } x \notin dom(\Gamma)$$

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$$(:_{\mathrm{app}}) \quad \frac{\Gamma \vdash M_1 : \tau \to \tau' \quad \Gamma \vdash M_2 : \tau}{\Gamma \vdash M_1\, M_2 : \tau'}$$

$$(:_{\mathrm{fix}}) \quad \frac{\Gamma \vdash M : \tau \to \tau}{\Gamma \vdash \mathbf{fix}(M) : \tau}$$

Partial recursive functions in PCF

- Primitive recursion.

p. rec.

$$\begin{cases} h(x,0) = f(x) \\ h(x,y+1) = g(x,y,h(x,y)) \end{cases}$$

$F : nat \to nat$

$H : nat \to nat \to nat$

$G : nat \to nat \to nat \to nat$

$H\ x\ y = \underline{\text{if}}\ \underline{\text{zero}}(y)\ \underline{\text{then}}\ F(x)$
$\quad \underline{\text{else}}\ G\ x\ (pred\ y)\ (H\ x\ (pred\ y))$

$(h\ x(pred\ y))$

$H =^{def} \underline{fix}(\lambda h. \lambda x. \lambda y. \text{if}\ \underline{\text{zero}}(y)\ \text{then}\ Fx\ \underline{\text{else}}\ G\ x(pred\ y)$

Partial recursive functions in PCF

- Primitive recursion.

$$\begin{cases} h(x,0) = f(x) \\ h(x,y+1) = g(x,y,h(x,y)) \end{cases}$$

- Minimisation.

$K : nat \to nat \to nat$

$$m(x) \;=\; \text{the least } y \geq 0 \text{ such that } k(x,y) = 0$$

H x y = if zero (K x y) then y
else H x (succ y)

H = fix (λh. λx. λy. ....)   M = fn x. H x 0

PCF evaluation relation

takes the form

$$M \Downarrow_\tau V$$

where

- τ is a PCF type
- $M, V \in \mathrm{PCF}_\tau$ are closed PCF terms of type τ
- V is a value,

$$V ::= \mathbf{0} \mid \mathbf{succ}(V) \mid \mathbf{true} \mid \mathbf{false} \mid \mathbf{fn}\, x : \tau \,.\, M.$$

PCF evaluation (sample rules)

$(\Downarrow_{\mathrm{val}})$ $\quad V \Downarrow_\tau V \qquad$ (V a value of type τ)

$$(\Downarrow_{\mathrm{cbn}}) \quad \frac{M_1 \Downarrow_{\tau \to \tau'} \mathbf{fn}\, x : \tau \,.\, M_1' \qquad M_1'[M_2/x] \Downarrow_{\tau'} V}{M_1\, M_2 \Downarrow_{\tau'} V}$$

PCF evaluation (sample rules)

$$(\Downarrow_{\mathrm{val}}) \quad V \Downarrow_\tau V \qquad (V \text{ a value of type } \tau)$$

$$(\Downarrow_{\mathrm{cbn}}) \quad \frac{M_1 \Downarrow_{\tau \to \tau'} \mathbf{fn}\, x : \tau \,.\, M_1' \qquad M_1'[M_2/x] \Downarrow_{\tau'} V}{M_1\, M_2 \Downarrow_{\tau'} V}$$

$$(\Downarrow_{\mathrm{fix}}) \quad \frac{M\, \mathbf{fix}(M) \Downarrow_\tau V}{\mathbf{fix}(M) \Downarrow_\tau V}$$

There is no V s.t. $\underline{fix}(fn\ x:\tau.\ x) \Downarrow V$

$$\left(fn\ x:\tau.\ x \mapsto D \xrightarrow[id]{} D \qquad fix(id) = \bot \right)$$

$$\dfrac{fn\ x:\tau.\ x \Downarrow fn\ x:\tau.\ x \qquad \dfrac{fix(fn\ x:\tau.\ x)}{x\left[{}^{fix(fn\ x:\tau.\ x)}/_{x}\right] \Downarrow \_\_}}{\dfrac{(fn\ x:\tau.\ x)\ (\underline{fix}(fn\ x:\tau.\ x)) \Downarrow \_\_}{\underline{fix}(fn\ x:\tau.\ x) \Downarrow \_\_}}$$

Contextual equivalence

Two phrases of a programming language are contextually equivalent if any occurrences of the first phrase in a complete program can be replaced by the second phrase without affecting the observable results of executing the program.

Contextual equivalence of PCF terms

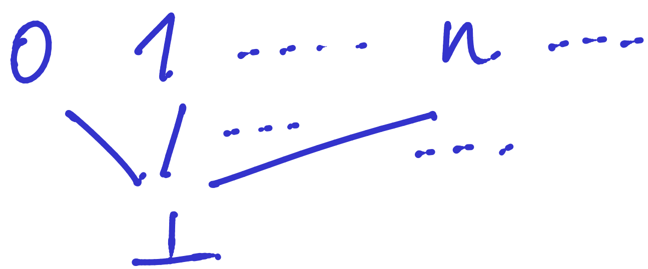
Given PCF terms M_1, M_2, PCF type τ, and a type environment Γ, the relation $\boxed{\Gamma \vdash M_1 \cong_{\mathrm{ctx}} M_2 : \tau}$ is defined to hold iff

- Both the typings $\Gamma \vdash M_1 : \tau$ and $\Gamma \vdash M_2 : \tau$ hold.
- For all PCF contexts $\mathcal{C}$ for which $\mathcal{C}[M_1]$ and $\mathcal{C}[M_2]$ are closed terms of type γ, *where* $\gamma = nat$ *or* $\gamma = bool$, and for all values $V : \gamma$,

$$\mathcal{C}[M_1] \Downarrow_\gamma V \;\Leftrightarrow\; \mathcal{C}[M_2] \Downarrow_\gamma V.$$

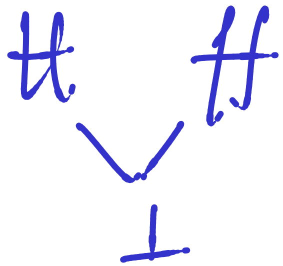
- PCF types $\tau \;\mapsto\;$ domains $[\![\tau]\!]$.

$[\![nat]\!] =$



$(n \in \mathbb{N})$

$[\![bool]\!] =$



$$[\![\tau_1 \to \tau_2]\!] = \left([\![\tau_1]\!] \to [\![\tau_2]\!]\right)$$

PCF denotational semantics — aims

- PCF types $\tau \;\mapsto\;$ domains $[\![\tau]\!]$.

- Closed PCF terms $M : \tau \;\mapsto\;$ elements $[\![M]\!] \in [\![\tau]\!]$.

  Denotations of open terms will be continuous functions.

$$\overbrace{x_1 : \tau_1,\ x_2 : \tau_2,\ \ldots,\ x_n : \tau_n}^{\Gamma} \vdash M : \tau$$

$$\underbrace{[\![\tau_1]\!] \times [\![\tau_2]\!] \times \cdots \times [\![\tau_n]\!] \longrightarrow [\![\tau]\!]}_{[\![\Gamma \vdash M : \tau]\!]}$$

PCF denotational semantics — aims

- PCF types $\tau \ \mapsto\ $ domains $[\![\tau]\!]$.
- Closed PCF terms $M : \tau \ \mapsto\ $ elements $[\![M]\!] \in [\![\tau]\!]$.

  Denotations of open terms will be continuous functions.
- Compositionality.

  In particular: $[\![M]\!] = [\![M']\!] \ \Rightarrow\ [\![\mathcal{C}[M]]\!] = [\![\mathcal{C}[M']]\!]$.
- Soundness.

  For any type τ, $M \Downarrow_\tau V \ \Rightarrow\ [\![M]\!] = [\![V]\!]$.
- Adequacy.

  For $\tau = bool$ or nat, $[\![M]\!] = [\![V]\!] \in [\![\tau]\!] \implies M \Downarrow_\tau V$.

Theorem. *For all types* τ *and closed terms* $M_1, M_2 \in \mathrm{PCF}_\tau$, *if* $[\![M_1]\!]$ *and* $[\![M_2]\!]$ *are equal elements of the domain* $[\![\tau]\!]$, *then* $M_1 \cong_{\mathrm{ctx}} M_2 : \tau$.

$$C[M_1] \Downarrow V \Rightarrow [\![C[M_1]]\!] = [\![V]\!]$$

$$\Rightarrow [\![C[M_2]]\!] = [\![V]\!]$$

$$\Rightarrow C[M_2] \Downarrow V$$

Theorem. *For all types τ and closed terms $M_1, M_2 \in \mathrm{PCF}_\tau$, if $[\![M_1]\!]$ and $[\![M_2]\!]$ are equal elements of the domain $[\![\tau]\!]$, then $M_1 \cong_{\mathrm{ctx}} M_2 : \tau$.*

Proof.

$$\mathcal{C}[M_1] \Downarrow_{nat} V \Rightarrow [\![\mathcal{C}[M_1]]\!] = [\![V]\!] \quad \text{(soundness)}$$

$$\Rightarrow [\![\mathcal{C}[M_2]]\!] = [\![V]\!] \quad \text{(compositionality on } [\![M_1]\!] = [\![M_2]\!]\text{)}$$

$$\Rightarrow \mathcal{C}[M_2] \Downarrow_{nat} V \quad \text{(adequacy)}$$

and symmetrically. $\square$

Proof principle

To prove

$$M_1 \cong_{\mathrm{ctx}} M_2 : \tau$$

it suffices to establish

$$[\![M_1]\!] = [\![M_2]\!] \text{ in } [\![\tau]\!]$$

$$\boxed{\dfrac{[\![M_1]\!] = [\![M_2]\!]}{M_1 \cong M_2}}$$

Proof principle

To prove

$$M_1 \cong_{\mathrm{ctx}} M_2 : \tau$$

it suffices to establish

$$[\![M_1]\!] = [\![M_2]\!] \text{ in } [\![\tau]\!]$$

$\boxed{?}$ The proof principle is sound, but is it complete? That is, is equality in the denotational model also a necessary condition for contextual equivalence?