Continuity of the fixpoint operator

Let D be a domain.

By Tarski's Fixed Point Theorem we know that each continuous function $f \in (D \to D)$ possesses a least fixed point, $\mathit{fix}(f) \in D$.

Proposition. *The function*

$$\mathit{fix} : (D \to D) \to D$$

is continuous.

$$\mathrm{fix} = \lambda f : D \to D.\ \bigsqcup_{n \geq 0} f^n(\bot)$$

(1) Monotonicity

$$f \sqsubseteq g \overset{?}{\Longrightarrow} \mathrm{fix}(f) \sqsubseteq \mathrm{fix}(g)$$

def $\Updownarrow$ $\qquad\qquad$ $\Updownarrow$ def

$$\forall d \in D.\ f(d) \sqsubseteq g(d) \qquad\qquad \bigsqcup_n f^n(\bot) \sqsubseteq \bigsqcup_n g^n(\bot)$$

$\surd$ by induction

$$\frac{\forall n \quad f^n(\bot) \sqsubseteq g^n(\bot)}{\bigsqcup_n f^n(\bot) \sqsubseteq \bigsqcup_n g^n(\bot)}$$

(2) For $f_0 \sqsubseteq f_1 \sqsubseteq \cdots \sqsubseteq f_n \sqsubseteq \cdots \quad (n \in \mathbb{N})$ in $(D \to D)$

Then $\mathrm{fix}\left(\bigsqcup_n f_n\right) \overset{?}{=} \bigsqcup_n \mathrm{fix}(f_n)$

$$\mathrm{fix}\left(\bigsqcup_n f_n\right) \overset{\mathrm{def}}{=} \bigsqcup_k \left(\bigsqcup_n f_n\right)^k (\bot)$$

$$\bigsqcup_n \mathrm{fix}(f_n) \overset{\mathrm{def}}{=} \bigsqcup_n \bigsqcup_k f_n^k(\bot)$$

BY INDUCTION

$$\bigsqcup_k \left(\bigsqcup_n f_n\right)^k (\bot) = \bigsqcup_k f_k^k(\bot) = \bigsqcup_n \bigsqcup_k f_n^k(\bot)$$

$$\left(\bigsqcup_n f_n\right)\left(\left(\bigsqcup_n f_n\right)(\bot)\right)$$

$$= \left(\bigsqcup_n f_n\right)\left(\bigsqcup_n (f_n(\bot))\right)$$

$$= \bigsqcup_m \left(f_m\left(\bigsqcup_n f_n(\bot)\right)\right) = \bigsqcup_m \bigsqcup_n f_m(f_n \bot)$$

$$\bigsqcup_k f_k(f_k \bot) \;=\; \bigsqcup_m \bigsqcup_n f_m(f_n \bot)$$

Topic 4

Scott Induction

Scott's Fixed Point Induction Principle

Let $f : D \to D$ be a continuous function on a domain D.

For any admissible subset $S \subseteq D$, to prove that the least fixed point of f is in S, *i.e.* that

$$fix(f) \in S \ ,$$

$fix(f) = \bigsqcup_n f^n(\bot)$

$\bigsqcup_n f^n(\bot) \in S$?

it suffices to prove

$$\forall d \in D \ (d \in S \ \Rightarrow \ f(d) \in S) \ .$$

$$(\bot \in S) \wedge \left(\frac{d_0 \sqsubseteq d_1 \sqsubseteq \cdots \sqsubseteq d_n \sqsubseteq \cdots \in S}{\bigsqcup_n d_n \in S} \right)$$

Chain-closed and admissible subsets

Let D be a cpo. A subset $S \subseteq D$ is called chain-closed iff for all chains $d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \ldots$ in D

$$(\forall n \geq 0 \,.\, d_n \in S) \;\Rightarrow\; \Big(\bigsqcup_{n \geq 0} d_n\Big) \in S$$

If D is a domain, $S \subseteq D$ is called admissible iff it is a chain-closed subset of D and $\perp \in S$.

$$\frac{\forall d \in D \,.\, d \in S \Rightarrow f(d) \in S}{fix(f) \in S} \;(S \text{ admissible})$$

Building chain-closed subsets (I)

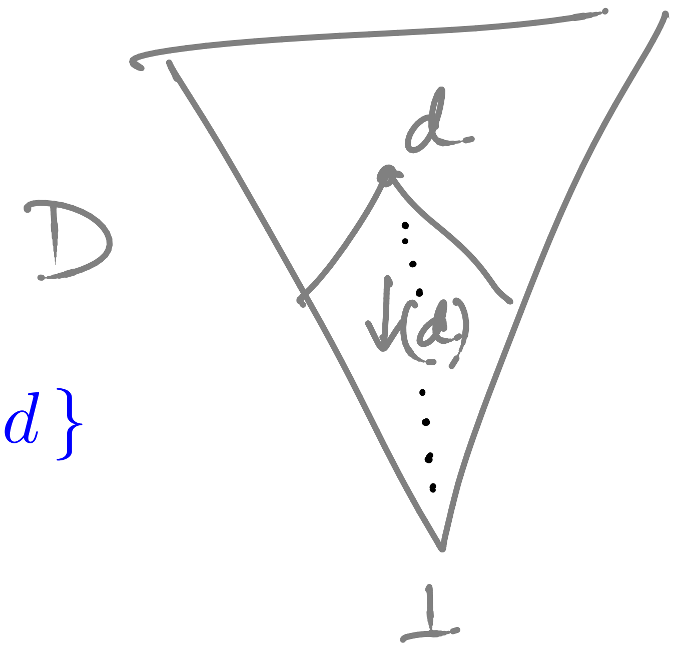
Let D, E be cpos.

Basic relations:

- For every $d \in D$, the subset

$$\downarrow(d) \stackrel{\text{def}}{=} \{\, x \in D \mid x \sqsubseteq d \,\}$$

of D is chain-closed.



Building chain-closed subsets (I)

Let D, E be cpos.

Basic relations:

- For every $d \in D$, the subset

$$\downarrow(d) \stackrel{\text{def}}{=} \{\, x \in D \mid x \sqsubseteq d \,\}$$ $y_0 \sqsubseteq y_1 \sqsubseteq \cdots \sqsubseteq y_n \sqsubseteq \cdots$

$x_0 \sqsubseteq x_1 \sqsubseteq \cdots \sqsubseteq x_n \sqsubseteq \cdots$

  of D is chain-closed.

- The subsets

$(x_0, y_0) \sqsubseteq \cdots \sqsubseteq (x_n, y_n) \sqsubseteq \cdots$ s.t. $x_i \sqsubseteq y_i$

$\Downarrow ?$

$$\{(x, y) \in D \times D \mid x \sqsubseteq y\}$$

  and

$$\{(x, y) \in D \times D \mid x = y\}$$

$\bigsqcup_n (x_n, y_n) = (\bigsqcup_n x_n, \bigsqcup_n y_n)$

$\bigsqcup_n x_n \sqsubseteq \bigsqcup_n y_n$

  of $D \times D$ are chain-closed.

Example (I): Least pre-fixed point property

Let D be a domain and let $f : D \to D$ be a continuous function.

$$\forall d \in D.\, f(d) \sqsubseteq d \implies \mathit{fix}(f) \sqsubseteq d$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[x \sqsubseteq d]}{f\,x \sqsubseteq f(d)}\; f \text{ mon} \qquad f(d) \sqsubseteq d}{f(x) \sqsubseteq d}}{}}{x \sqsubseteq d \Rightarrow f(x) \sqsubseteq d}}{\mathit{fix}(f) \sqsubseteq d}$$

$\downarrow(d) = \{x \mid x \sqsubseteq d\}$ admissible

Example (I): Least pre-fixed point property

Let D be a domain and let $f : D \to D$ be a continuous function.

$$\forall d \in D.\, f(d) \sqsubseteq d \implies \mathit{fix}(f) \sqsubseteq d$$

Proof by Scott induction.

Let $d \in D$ be a pre-fixed point of f. Then,

$$\begin{aligned} x \in {\downarrow}(d) &\implies x \sqsubseteq d \\ &\implies f(x) \sqsubseteq f(d) \\ &\implies f(x) \sqsubseteq d \\ &\implies f(x) \in {\downarrow}(d) \end{aligned}$$

Hence,

$$\mathit{fix}(f) \in {\downarrow}(d) \ .$$

$x_0 \sqsubseteq x_1 \sqsubseteq \cdots \sqsubseteq x_n \sqsubseteq \cdots \in f^{-1}(S)$

$f(x_0), f(x_1), \cdots f(x_n) \cdots \in S \Rightarrow \bigsqcup_n f x_n \in S$

Building chain-closed subsets (II)

Inverse image:

RTP: $\bigsqcup_n x_n \in f^{-1}(S)$

Let $f : D \to E$ be a continuous function. $\Leftrightarrow$ $f(\bigsqcup_n x_n) \in S$

If S is a chain-closed subset of E then the inverse image

$$f^{-1}S = \{x \in D \mid f(x) \in S\}$$

is an chain-closed subset of D.

$f(\bigsqcup_n x_n) = \bigsqcup_n f(x_n)$

Example (II)

Let D be a domain and let $f, g : D \to D$ be continuous functions such that $f \circ g \sqsubseteq g \circ f$. Then,

$$f(\bot) \sqsubseteq g(\bot) \implies \mathit{fix}(f) \sqsubseteq \mathit{fix}(g) \ .$$

$S = \{ x \mid f(x) \sqsubseteq g(x) \}$ admissible

$\langle f, g \rangle : D \to D \times D : x \mapsto (fx, gx)$

cont.

$\sqsubseteq$ admissible

$\langle f, g \rangle^{-1}(\sqsubseteq)$ admissible

$\parallel$

S

$$f \circ g \sqsubseteq g \circ f \qquad\qquad f(\bot) \sqsubseteq g(\bot)$$

$$\bigsqcup_n g^n \bot$$

$$\bot \sqsubseteq g\bot \sqsubseteq g^2\bot \sqsubseteq \cdots \qquad g^n(\bot) \sqsubseteq \cdots \sqsubseteq \mathrm{fix}\, g$$

$$g^2\bot \sqsupseteq g f \bot \sqsupseteq f g \bot \sqsupseteq f^2(\bot)$$

by induction

$$\bot \sqsubseteq f\bot \sqsubseteq f^2(\bot) \sqsubseteq \cdots \sqsubseteq f^n\bot \cdots \sqsubseteq \mathrm{fix}\, f$$

$$\mathrm{fix}\, f = \bigsqcup_n f^n \bot$$

$$\mathrm{fix}(f) \sqsubseteq \mathrm{fix}(g)$$

$$\dfrac{\dfrac{\text{HYP}}{fgx \sqsubseteq gfx} \qquad \begin{array}{c}[fx \sqsubseteq gx] \\ gfx \sqsubseteq ggx\end{array}}{fgx \sqsubseteq ggx}$$

$$\dfrac{\dfrac{\cdots}{f(x) \sqsubseteq g(x) \Rightarrow f(gx) \sqsubseteq g(gx)}}{f(\mathrm{fix}\, g) \sqsubseteq g(\mathrm{fix}\, g)}\ \delta I$$

$$\dfrac{\dfrac{f(\mathrm{fix}\, g) \sqsubseteq g(\mathrm{fix}\, g)}{f(\mathrm{fix}(g)) \sqsubseteq \mathrm{fix}(g)}}{\mathrm{fix}(f) \sqsubseteq \mathrm{fix}(g)}$$

Example (II)

Let D be a domain and let $f, g : D \to D$ be continuous functions such that $f \circ g \sqsubseteq g \circ f$. Then,

$$f(\bot) \sqsubseteq g(\bot) \implies \mathit{fix}(f) \sqsubseteq \mathit{fix}(g) \ .$$

Proof by Scott induction.

Consider the admissible property $\Phi(x) \equiv \big(f(x) \sqsubseteq g(x)\big)$ of D.

Since

$$f(x) \sqsubseteq g(x) \Rightarrow g(f(x)) \sqsubseteq g(g(x)) \Rightarrow f(g(x)) \sqsubseteq g(g(x))$$

we have that

$$f(\mathit{fix}(g)) \sqsubseteq g(\mathit{fix}(g)) \ .$$

Building chain-closed subsets (III)

Logical operations:

- If $S, T \subseteq D$ are chain-closed subsets of D then

  $$S \cup T \qquad \text{and} \qquad S \cap T$$

  are chain-closed subsets of D.

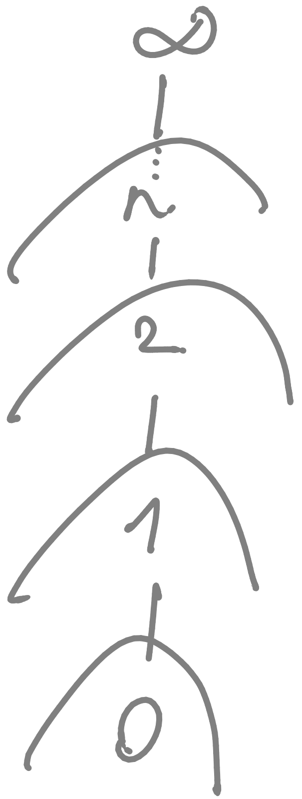
- If $\{ S_i \}_{i \in I}$ is a family of chain-closed subsets of D indexed by a set I, then $\bigcap_{i \in I} S_i$ is a chain-closed subset of D.

- If a property $P(x, y)$ determines a chain-closed subset of $D \times E$, then the property $\forall x \in D.\ P(x, y)$ determines a chain-closed subset of E.

Arbitrary unions of admissible subsets of a domain need not be admissible

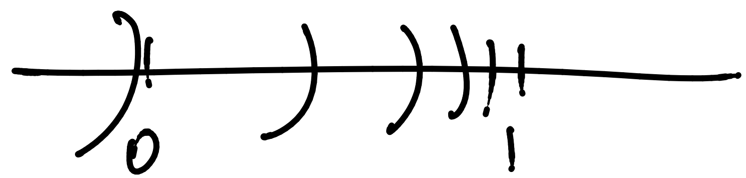
$S_i \subseteq D$ $\qquad$ S_i admissible

but $\bigcup_i S_i$ not admissible



$S_n = \downarrow(n)$

$\bigcup_n S_n = \mathbb{N}$



Example (III): Partial correctness

Let $\mathcal{F} : State \rightharpoonup State$ be the denotation of

$$\textbf{while } X > 0 \textbf{ do } (Y := X * Y; X := X - 1) \ .$$

For all $x, y \geq 0$,

$$\mathcal{F}[X \mapsto x, Y \mapsto y] \downarrow$$
$$\implies \mathcal{F}[X \mapsto x, Y \mapsto y] = [X \mapsto 0, Y \mapsto x! \cdot y].$$

Recall that

$$\mathcal{F} = \mathit{fix}(f)$$

where $f : (State \rightharpoonup State) \to (State \rightharpoonup State)$ is given by

$$f(w) = \lambda(x, y) \in State. \begin{cases} (x, y) & \text{if } x \leq 0 \\ w(x - 1, x \cdot y) & \text{if } x > 0 \end{cases}$$

Proof by Scott induction.

We consider the admissible subset of $(State \rightharpoonup State)$ given by

$$S = \left\{ w \;\middle|\; \begin{array}{l} \forall x, y \geq 0. \\ \quad w[X \mapsto x, Y \mapsto y]\downarrow \\ \qquad \Rightarrow w[X \mapsto x, Y \mapsto y] = [X \mapsto 0, Y \mapsto x! \cdot y] \end{array} \right\}$$

and show that

$$w \in S \implies f(w) \in S \;.$$