Continuity and strictness

- If D and E are cpo's, the function f is continuous iff
  1. it is monotone, and
  2. it preserves lubs of chains, *i.e.* for all chains $d_0 \sqsubseteq d_1 \sqsubseteq \ldots$ in D, it is the case that

$$f(\bigsqcup_{n \geq 0} d_n) = \bigsqcup_{n \geq 0} f(d_n) \quad \text{in } E.$$

- If D and E have least elements, then the function f is strict iff $f(\perp) = \perp$.

Tarski's Fixed Point Theorem

Let $f : D \to D$ be a continuous function on a domain D. Then

- f possesses a least pre-fixed point, given by

$$\mathit{fix}(f) = \bigsqcup_{n \geq 0} f^n(\bot).$$

- Moreover, $\mathit{fix}(f)$ is a fixed point of f, *i.e.* satisfies $f\big(\mathit{fix}(f)\big) = \mathit{fix}(f)$, and hence is the least fixed point of f.

For continuous $f : D \to D$

$$fix(f) \stackrel{def}{=} \bigsqcup_n f^n(\bot)$$

is a least ^{pre}fixed point.

NB: $\bot, f(\bot), f(f\bot), \ldots, f^n(\bot), \ldots$ is a chain!

$$\frac{\bot \sqsubseteq f(\bot)}{\frac{f(\bot) \sqsubseteq ff\bot}{\frac{ff\bot \sqsubseteq fff\bot}{\ldots}}}$$

(1) It is a pre fixed point.

I.e. $f(fix(f)) \sqsubseteq fix(f)$

$$f\Big(\bigsqcup_{n\geq 0} f^n(\bot)\Big) = \bigsqcup_{n\geq 0} f(f^n \bot) = \bigsqcup_{n\geq 0} f^{n+1}(\bot)$$

$$= \bigsqcup_{n\geq 1} f^n(\bot) = \bigsqcup_{n\geq 0} f^n(\bot) = fix(f)$$

(2) Say d is such that $f(d) \sqsubseteq d$

$\boxed{?}$ $\mathrm{fix}(f) \sqsubseteq d$?

$\perp \sqsubseteq d$ ✓ $\Rightarrow f\perp \sqsubseteq fd \sqsubseteq d$

$\Rightarrow ff\perp \sqsubseteq fd \sqsubseteq d$

$\Rightarrow \cdots$

$\perp \sqsubseteq d$ ✓

$f(\perp) \sqsubseteq d$ ✓

$ff(\perp) \sqsubseteq d$

$\vdots$

By induction! ✓

$$\frac{\dfrac{}{\forall n \quad f^n(\perp) \sqsubseteq d}}{\mathrm{fix}(f) = \bigsqcup_n f^n(\perp) \sqsubseteq d}$$

$[\![\textbf{while } B \textbf{ do } C]\!]$

$$[\![\textbf{while } B \textbf{ do } C]\!]$$

$$= \mathit{fix}(f_{[\![B]\!],[\![C]\!]})$$

$$= \bigsqcup_{n \geq 0} {f_{[\![B]\!],[\![C]\!]}}^{n}(\perp)$$

$$= \quad \lambda s \in \mathit{State}.$$

$$\begin{cases} [\![C]\!]^k(s) & \text{if } k \geq 0 \text{ is such that } [\![B]\!]([\![C]\!]^k(s)) = \mathit{false} \\ & \text{and } [\![B]\!]([\![C]\!]^i(s)) = \mathit{true} \text{ for all } 0 \leq i < k \\ \\ \text{undefined} & \text{if } [\![B]\!]([\![C]\!]^i(s)) = \mathit{true} \text{ for all } i \geq 0 \end{cases}$$

$f_{[\![B]\!],[\![C]\!]} = \lambda w. \lambda s.$
$\quad \mathit{if}([\![B]\!]s, w([\![C]\!]s), s)$

is continuous $(\mathit{State} \to \mathit{State}) \to (\mathit{State} \to \mathit{State})$

What about domains of computation for data type constructions?

α, β types

$\alpha * \beta$ type

$\alpha \to \beta$ type

recursive datatypes; e.g. lists, trees, ...

Topic 3

Constructions on Domains

Discrete cpo's and flat domains

For any set X, the relation of equality

$$x \sqsubseteq x' \overset{\text{def}}{\Leftrightarrow} x = x' \qquad (x, x' \in X)$$

makes $(X, \sqsubseteq)$ into a cpo, called the discrete cpo with underlying set X.

Let $X_\perp \overset{\text{def}}{=} X \cup \{\perp\}$, where $\perp$ is some element not in X. Then

$$d \sqsubseteq d' \overset{\text{def}}{\Leftrightarrow} (d = d') \vee (d = \perp) \qquad (d, d' \in X_\perp)$$

makes $(X_\perp, \sqsubseteq)$ into a domain (with least element $\perp$), called the flat domain determined by X.

Products

D, E domains.

$D \times E$ ~ underlying set

$\{(d, e) \mid d \in D \wedge e \in E\}$

~ partial order

$(d_1, e_1) \sqsubseteq_{D \times E} (d_2, e_2)$

$\overset{\text{def}}{\iff}$ $d_1 \sqsubseteq_D d_2 \wedge e_1 \sqsubseteq_E e_2$

<u>Check</u>:

$\sqsubseteq_{D \times E}$ is a poset with least element and least upper bounds of chains.

Binary product of cpo's and domains

The product of two cpo's $(D_1, \sqsubseteq_1)$ and $(D_2, \sqsubseteq_2)$ has underlying set

$$D_1 \times D_2 = \{(d_1, d_2) \mid d_1 \in D_1 \ \& \ d_2 \in D_2\}$$

and partial order $\sqsubseteq$ defined by

$$(d_1, d_2) \sqsubseteq (d_1', d_2') \overset{\text{def}}{\Leftrightarrow} d_1 \sqsubseteq_1 d_1' \ \& \ d_2 \sqsubseteq_2 d_2' \ .$$

$$\frac{(x_1, x_2) \sqsubseteq (y_1, y_2)}{x_1 \sqsubseteq_1 y_1 \qquad x_2 \sqsubseteq_2 y_2}$$

A chain in $D_1 \times D_2$

$(d_{1,0}, d_{2,0}) \sqsubseteq (d_{1,1}, d_{2,1}) \sqsubseteq \cdots \sqsubseteq (d_{1,n}, d_{2,n}) \sqsubseteq \cdots$

$d_{1,0} \sqsubseteq d_{1,1} \sqsubseteq \cdots \sqsubseteq d_{1,n} \sqsubseteq \cdots$ in D_1

$d_{2,0} \sqsubseteq d_{2,1} \sqsubseteq \cdots \sqsubseteq d_{2,n} \sqsubseteq \cdots$ in D_2

$\bigsqcup_{n \geq 0} (d_{1,n}, d_{2,n})$

Lubs of chains are calculated componentwise:

$$\bigsqcup_{n \geq 0} (d_{1,n}, d_{2,n}) = \left(\bigsqcup_{i \geq 0} d_{1,i}, \bigsqcup_{j \geq 0} d_{2,j}\right) \ .$$

If $(D_1, \sqsubseteq_1)$ and $(D_2, \sqsubseteq_2)$ are domains so is $(D_1 \times D_2, \sqsubseteq)$ and $\perp_{D_1 \times D_2} = (\perp_{D_1}, \perp_{D_2})$.

$\bigsqcup_n d_{2,n}$ in D_2

$\bigsqcup_n d_{1,n}$ in D_1

$\rightsquigarrow \left(\bigsqcup_n d_{1,n}, \bigsqcup_n d_{2,n}\right)$ in $D_1 \times D_2$

Continuous functions of two arguments

Proposition. *Let D, E, F be cpo's. A function $f : (D \times E) \to F$ is monotone if and only if it is monotone in each argument separately:*

$$\forall d, d' \in D, e \in E.\, d \sqsubseteq d' \;\Rightarrow\; f(d,e) \sqsubseteq f(d',e)$$ e fixed

$$\forall d \in D, e, e' \in E.\, e \sqsubseteq e' \;\Rightarrow\; f(d,e) \sqsubseteq f(d,e').$$ d fixed

Moreover, it is continuous if and only if it preserves lubs of chains in each argument separately:

e fixed

$$f(\bigsqcup_{m \geq 0} d_m\,,\, e) = \bigsqcup_{m \geq 0} f(d_m, e)$$

d fixed

$$f(d\,,\, \bigsqcup_{n \geq 0} e_n) = \bigsqcup_{n \geq 0} f(d, e_n).$$

- A couple of derived rules:

$$\frac{x \sqsubseteq x' \qquad y \sqsubseteq y'}{f(x,y) \sqsubseteq f(x',y')} \quad (f \text{ monotone})$$

$$\frac{}{f(\bigsqcup_m x_m, \bigsqcup_n y_n) = \bigsqcup_k f(x_k, y_k)} \quad (f \text{ continuous})$$

Because

$$f(\bigsqcup_m x_m, \bigsqcup_n y_n) = \bigsqcup_m f(x_m, \bigsqcup_n y_n) = \bigsqcup_m \bigsqcup_n f(x_m, y_n)$$

// diagonalisation lemma.

Function cpo's and domains

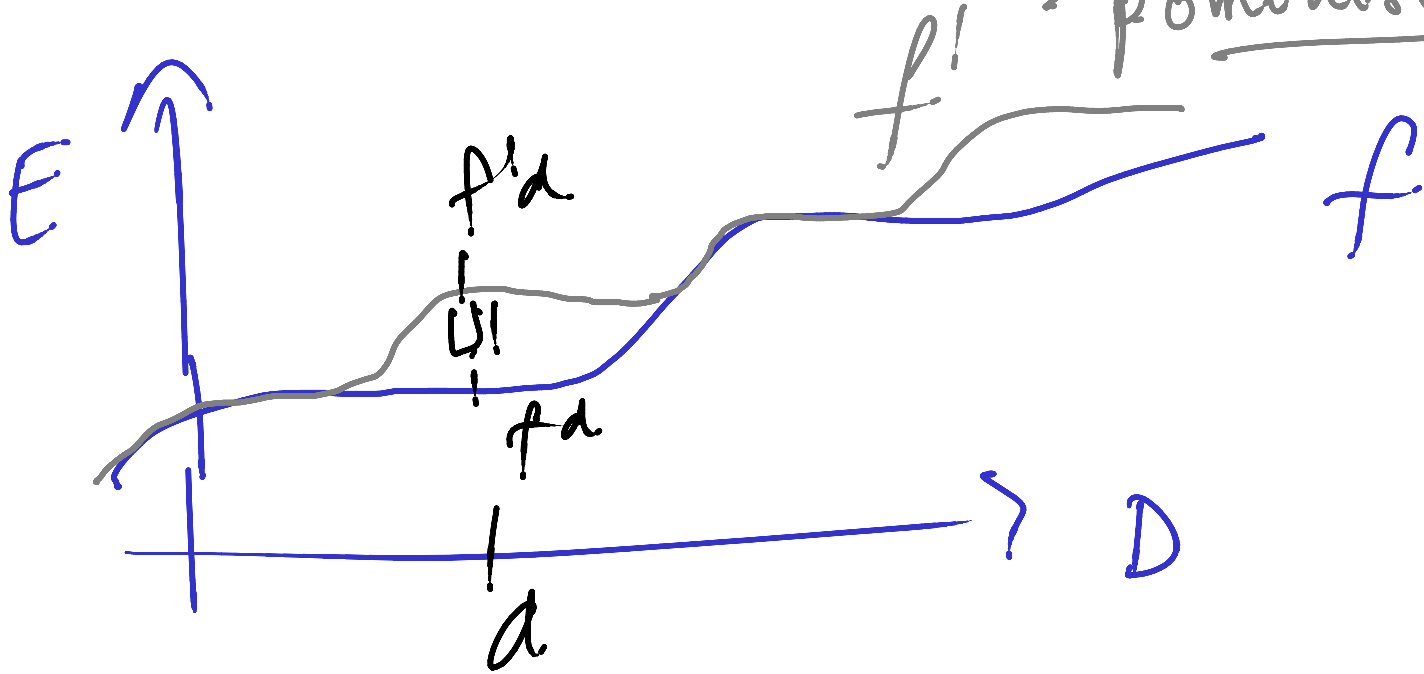
Given cpo's $(D, \sqsubseteq_D)$ and $(E, \sqsubseteq_E)$, the function cpo $(D \to E, \sqsubseteq)$ has underlying set

$$(D \to E) \stackrel{\text{def}}{=} \{f \mid f : D \to E \text{ is a } \textit{continuous} \text{ function}\}$$

and partial order: $f \sqsubseteq f' \stackrel{\text{def}}{\Leftrightarrow} \forall d \in D \,.\, f(d) \sqsubseteq_E f'(d)$.

$(D \to E)$

pointwise



Check

$\sqsubseteq_{D \to E}$ is a poset

has lubs of chains.

$f_0 \sqsubseteq f_1 \sqsubseteq \cdots \sqsubseteq f_n \sqsubseteq \cdots \quad \bigsqcup_n f_n$ in $(D \to E)$

$\underbrace{}_{\text{a continuous function } D \to E}$

(1) Define

$$\left(\bigsqcup_n f_n\right)(d) \stackrel{def}{=} \bigsqcup_n \left(f_n(d)\right)$$

$f_0(d) \sqsubseteq f_1(d) \sqsubseteq \cdots f_n(d) \sqsubseteq \cdots$ in E

$\bigsqcup_n f_n(d)$ in E

(2) $\bigsqcup_n f_n$ is continuous.

(i) monotone

$$d \sqsubseteq_D d' \Longrightarrow \left(\bigsqcup_n f_n\right)(d) \sqsubseteq \left(\bigsqcup_n f_n\right)(d')$$

$$\left(\bigsqcup_n f_n\right)(d) = \bigsqcup_n \left(f_n(d)\right) \qquad \left(\bigsqcup_n f_n\right)(d') = \bigsqcup_n \left(f_n(d')\right)$$

$$\Uparrow$$

$$f_n(d) \sqsubseteq f_n(d')$$

$$\Uparrow \quad f_n \text{ monotone}$$

$$d \sqsubseteq d'$$

(ii) $\bigsqcup_n f_n$ preserves lubs of chains.

Consider $d_0 \sqsubseteq d_1 \sqsubseteq \cdots \sqsubseteq d_n \sqsubseteq \cdots$ in D

$$\left(\bigsqcup_n f_n\right)\left(\bigsqcup_m d_m\right) \stackrel{?}{=} \bigsqcup_m \left(\bigsqcup_n f_n\right)(d_m)$$

def $\parallel$ $\qquad\qquad$ $\parallel$ def

$$\bigsqcup_n f_n\left(\bigsqcup_m d_m\right) \qquad \bigsqcup_m \bigsqcup_n f_n(d_m)$$

$\parallel$ f_n cont

$$\bigsqcup_n \bigsqcup_m f_n(d_m) \;=\!=\; \text{diag. lemma.}$$

Function cpo's and domains

Given cpo's $(D, \sqsubseteq_D)$ and $(E, \sqsubseteq_E)$, the function cpo $(D \to E, \sqsubseteq)$ has underlying set

$$(D \to E) \stackrel{\text{def}}{=} \{f \mid f : D \to E \text{ is a } \textit{continuous} \text{ function}\}$$

and partial order: $f \sqsubseteq f' \stackrel{\text{def}}{\Leftrightarrow} \forall d \in D \,.\, f(d) \sqsubseteq_E f'(d)$.

- A derived rule:

$$\frac{f \sqsubseteq_{(D \to E)} g \qquad x \sqsubseteq_D y}{f(x) \sqsubseteq g(y)}$$

Lubs of chains are calculated 'argumentwise' (using lubs in E):

$$\bigsqcup_{n\geq 0} f_n = \lambda d \in D. \bigsqcup_{n\geq 0} f_n(d) \ .$$

NB: P poset, D cpo/domain

$\Rightarrow$ $(P \to_{mon} D)$ cpo/domain

the monotone functions

If E is a domain, then so is $D \to E$ and $\perp_{D\to E}(d) = \perp_E$, all $d \in D$.

Ex: $P = (0 \leq 1 \leq 2 \leq \cdots \leq n \leq \cdots)$ $n \in \mathbb{N}$

$(P \to_{mon} D) = Ch(D)$

Lubs of chains are calculated 'argumentwise' (using lubs in E):

$$\bigsqcup_{n\geq 0} f_n = \lambda d \in D. \bigsqcup_{n\geq 0} f_n(d) \ .$$

- A derived rule:

$$\frac{}{(\bigsqcup_n f_n)(\bigsqcup_m x_m) = \bigsqcup_k f_k(x_k)}$$

If E is a domain, then so is $D \to E$ and $\perp_{D\to E}(d) = \perp_E$, all $d \in D$.

$\underline{\underline{ML}}$ $fn\ (g, f) \Rightarrow fn\ x \Rightarrow g(f(x))$

Continuity of composition

For cpo's D, E, F, the composition function

$$\circ : \big((E \to F) \times (D \to E)\big) \longrightarrow (D \to F)$$

defined by setting, for all $f \in (D \to E)$ and $g \in (E \to F)$,

$$g \circ f \;=\; \lambda d \in D.\, g\big(f(d)\big)$$

is continuous.