Pre-fixed points

Let D be a poset and $f : D \to D$ be a monotone function.

An element $d \in D$ is a pre-fixed point of f if it satisfies $f(d) \sqsubseteq d$.

The *least pre-fixed point* of f, if it exists, will be written

$$\boxed{\mathit{fix}(f)}$$

It is thus (uniquely) specified by the two properties:

$$f(\mathit{fix}(f)) \sqsubseteq \mathit{fix}(f) \qquad \text{(lfp1)}$$

$$\forall d \in D.\ f(d) \sqsubseteq d \Rightarrow \mathit{fix}(f) \sqsubseteq d. \qquad \text{(lfp2)}$$

Proof principle

2. Let D be a poset and let $f : D \to D$ be a function with a least pre-fixed point $\mathit{fix}(f) \in D$.

   For all $x \in D$, to prove that $\mathit{fix}(f) \sqsubseteq x$ it is enough to establish that $f(x) \sqsubseteq x$.

Proof Principle

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1.

$$\frac{}{f(\mathit{fix}(f)) \sqsubseteq \mathit{fix}(f)}$$

2. Let D be a poset and let $f : D \to D$ be a function with a least pre-fixed point $\mathit{fix}(f) \in D$.

   For all $x \in D$, to prove that $\mathit{fix}(f) \sqsubseteq x$ it is enough to establish that $f(x) \sqsubseteq x$.

$$\frac{f(x) \sqsubseteq x}{\mathit{fix}(f) \sqsubseteq x}$$

Least pre-fixed points are fixed points

If it exists, the least pre-fixed point of a mononote function on a partial order is necessarily a fixed point.

$$
\dfrac{\dfrac{\checkmark}{f(\mathrm{fix}\ f) \sqsubseteq \mathrm{fix}\ f} \qquad \dfrac{?}{\mathrm{fix}(f) \sqsubseteq f(\mathrm{fix}\ f)}}{f(\mathrm{fix}\ f) = \mathrm{fix}(f)}
$$

$$
\dfrac{\dfrac{\dfrac{\checkmark}{f(\mathrm{fix}\ f) \sqsubseteq \mathrm{fix}(f)}}{f(f(\mathrm{fix}\ f)) \sqsubseteq f(\mathrm{fix}\ f)}\ \text{fmon}}{\mathrm{fix}(f) \sqsubseteq f(\mathrm{fix}\ f)}\ \text{lfp2}
$$

Thesis*

All domains of computation are complete partial orders with a least element.

allows passage to the limit.

$d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \cdots \sqsubseteq d_n \sqsubseteq \cdots \qquad \bigsqcup_n d_n$

$f : D \to E$

$d_0 \sqsubseteq d_1 \sqsubseteq \cdots \sqsubseteq d_n \sqsubseteq \cdots \qquad \bigsqcup_n d_n$

$f(d_0) \sqsubseteq f(d_1) \sqsubseteq \cdots$ **Thesis*** $\sqsubseteq f(d_n) \sqsubseteq \cdots \qquad \bigsqcup_n f(d_n)$

▷ How do $f(\bigsqcup_n d_n)$ compare to $\bigsqcup_n f(d_n)$? ?

All domains of computation are
complete partial orders with a least element.

All computable functions are
continuous.

monotone + respects passage to the limit.

$d_0 \sqsubseteq d_1 \sqsubseteq \cdots \sqsubseteq d_n \sqsubseteq \cdots$

Cpo's and domains

A chain complete poset, or cpo for short, is a poset $(D, \sqsubseteq)$ in which all countable increasing chains $d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \ldots$ have least upper bounds, $\bigsqcup_{n \geq 0} d_n$:

$$\forall m \geq 0 \,.\, d_m \sqsubseteq \bigsqcup_{n \geq 0} d_n \tag{lub1}$$

$$\forall d \in D \,.\, (\forall m \geq 0 \,.\, d_m \sqsubseteq d) \;\Rightarrow\; \bigsqcup_{n \geq 0} d_n \sqsubseteq d. \tag{lub2}$$

A domain is a cpo that possesses a least element, $\bot$:

$$\forall d \in D \,.\, \bot \sqsubseteq d.$$

$$\frac{}{\perp \sqsubseteq x}$$

$$\frac{}{x_i \sqsubseteq \bigsqcup_{n \geq 0} x_n} \quad (i \geq 0 \text{ and } \langle x_n \rangle \text{ a chain})$$

$$\frac{\forall n \geq 0 \,.\, x_n \sqsubseteq x}{\bigsqcup_{n \geq 0} x_n \sqsubseteq x} \quad (\langle x_i \rangle \text{ a chain})$$

Domain of partial functions, $X \rightharpoonup Y$

Underlying set: all partial functions, f, with domain of definition $dom(f) \subseteq X$ and taking values in Y.

Partial order:

$$\begin{aligned} f \sqsubseteq g \quad & \text{iff} \quad dom(f) \subseteq dom(g) \text{ and} \\ & \qquad \forall x \in dom(f).\ f(x) = g(x) \\ & \text{iff} \quad graph(f) \subseteq graph(g) \end{aligned}$$

Lub of chain $f_0 \sqsubseteq f_1 \sqsubseteq f_2 \sqsubseteq \ldots$ is the partial function f with $dom(f) = \bigcup_{n \geq 0} dom(f_n)$ and

$$f(x) = \begin{cases} f_n(x) & \text{if } x \in dom(f_n), \text{ some } n \\ \text{undefined} & \text{otherwise} \end{cases}$$

$graph(f)$

$\|$

$graph(f_0) \subseteq \cdots \subseteq graph(f_n) \subseteq \cdots \subseteq \bigcup_n graph(f_n)$

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$$\quad \text{iff} \quad graph(f) \subseteq graph(g)$$

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$$f(x) = \begin{cases} f_n(x) & \text{if } x \in dom(f_n), \text{ some } n \\ \text{undefined} & \text{otherwise} \end{cases}$$

Least element $\perp$ is the totally undefined partial function.

$d \sqsubseteq d \sqsubseteq \cdots \sqsubseteq d \sqsubseteq \cdots$

Some properties of lubs of chains

Let D be a cpo.

1. For $d \in D$, $\bigsqcup_n d = d$.
2. For every chain $d_0 \sqsubseteq d_1 \sqsubseteq \ldots \sqsubseteq d_n \sqsubseteq \ldots$ in D,

$$\bigsqcup_n d_n = \bigsqcup_n d_{N+n}$$

for all $N \in \mathbb{N}$.

$d_0 \sqsubseteq d_1 \sqsubseteq \cdots \sqsubseteq d_N \sqsubseteq d_{N+1} \sqsubseteq \cdots$ $\quad \bigsqcup_n d_n$

$d_N \sqsubseteq d_{N+1} \sqsubseteq \cdots$ $\quad \bigsqcup_n d_{N+n}$

$$\begin{array}{ccccccccc}
e_0 \sqsubseteq & e_1 \sqsubseteq & \cdots & \sqsubseteq e_n \sqsubseteq & \cdots & & & \sqsubseteq & \bigsqcup_n e_n \\
\sqcup\!\!| & \sqcup\!\!| & \cdots & \sqcup\!\!| & & \cdots & & & \sqcup\!\!| \\
d_0 \sqsubseteq & d_1 \sqsubseteq & \cdots & \sqsubseteq d_n \sqsubseteq & \cdots & & & \sqsubseteq & \bigsqcup_n d_n
\end{array}$$

3. For every pair of chains $d_0 \sqsubseteq d_1 \sqsubseteq \ldots \sqsubseteq d_n \sqsubseteq \ldots$ and $e_0 \sqsubseteq e_1 \sqsubseteq \ldots \sqsubseteq e_n \sqsubseteq \ldots$ in D,

   if $d_n \sqsubseteq e_n$ for all $n \in \mathbb{N}$ then $\bigsqcup_n d_n \sqsubseteq \bigsqcup_n e_n$.

$$\dfrac{\dfrac{\dfrac{\checkmark}{d_n \sqsubseteq \underset{\sim}{e_n}}\,(\mathrm{hyp}) \qquad \dfrac{\checkmark}{\underset{\sim}{e_n} \sqsubseteq \bigsqcup_m e_m}\,(\mathrm{lub1})}{\forall n.\ d_n \sqsubseteq \bigsqcup_m e_m}\,(\mathrm{trans})}{\bigsqcup_n d_n \sqsubseteq \bigsqcup_m e_m}\,(\mathrm{lub2})$$

D domain — Ch(D) has elements countable chains in D, that is,
$(d_0 \sqsubseteq d_1 \sqsubseteq \dots \sqsubseteq d_n \sqsubseteq \dots)_{n\in\mathbb{N}}$

3. For every pair of chains $d_0 \sqsubseteq d_1 \sqsubseteq \ldots \sqsubseteq d_n \sqsubseteq \ldots$ and $e_0 \sqsubseteq e_1 \sqsubseteq \ldots \sqsubseteq e_n \sqsubseteq \ldots$ in D,

   if $d_n \sqsubseteq e_n$ for all $n \in \mathbb{N}$ then $\bigsqcup_n d_n \sqsubseteq \bigsqcup_n e_n$.

Ch(D) is a partial order with $(d_n)_n \sqsubseteq (e_n)_n \overset{\text{def}}{\iff} \forall n\in\mathbb{N}.\ d_n \sqsubseteq e_n$.

$$\frac{\forall n \geq 0\,.\,x_n \sqsubseteq y_n}{\bigsqcup_n x_n \sqsubseteq \bigsqcup_n y_n} \quad (\langle x_n\rangle \text{ and } \langle y_n\rangle \text{ chains})$$

$\bigsqcup : \mathrm{Ch}(D) \to D$

$(d_n)_{n\in\mathbb{N}} \mapsto (\bigsqcup_n d_n) \in D$

$\bigsqcup$ is monotone

Diagonalising a double chain

Lemma. *Let D be a cpo. Suppose that the doubly-indexed family of elements $d_{m,n} \in D$ ($m, n \geq 0$) satisfies*

$$m \leq m' \;\&\; n \leq n' \;\Rightarrow\; d_{m,n} \sqsubseteq d_{m',n'}. \qquad (\dagger)$$

Then

$$\bigsqcup_{n \geq 0} d_{0,n} \;\sqsubseteq\; \bigsqcup_{n \geq 0} d_{1,n} \;\sqsubseteq\; \bigsqcup_{n \geq 0} d_{2,n} \;\sqsubseteq\; \ldots$$

and

$$\bigsqcup_{m \geq 0} d_{m,0} \;\sqsubseteq\; \bigsqcup_{m \geq 0} d_{m,1} \;\sqsubseteq\; \bigsqcup_{m \geq 0} d_{m,3} \;\sqsubseteq\; \ldots$$

$$
\begin{array}{cccccccccc}
\bigsqcup_m d_0^{(m)} & \sqsubseteq & \bigsqcup_m d_1^{(m)} & \sqsubseteq & \cdots & & & & \bigsqcup_n \bigsqcup_m d_n^{(m)} \\
\rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & & & & & \| \\
\vdots & & \vdots & & & & & & \\
\rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & & & & & \\
d_0^{(m)} & \sqsubseteq & d_1^{(m)} & \sqsubseteq & \cdots & \sqsubseteq & d_n^{(m)} & \sqsubseteq \cdots & \bigsqcup_n d_n^{(m)} \\
\rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & & & \rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} \\
\vdots & & \vdots & & & & \vdots & & \vdots \\
\rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & & & & & \\
d_0^{(1)} & \sqsubseteq & d_1^{(1)} & \sqsubseteq & \cdots & \sqsubseteq & d_n^{(1)} & \sqsubseteq \cdots & \bigsqcup_n d_n^{(1)} \\
\rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} & & & & \rotatebox{90}{$\sqsubseteq$} & & \rotatebox{90}{$\sqsubseteq$} \\
d_0^{(0)} & \sqsubseteq & d_1^{(0)} & \sqsubseteq & \cdots & \sqsubseteq & d_n^{(0)} & \sqsubseteq \cdots & \bigsqcup_n d_n^{(0)}
\end{array}
$$

$$\bigsqcup_k d_k^{(k)} = \bigsqcup_m \bigsqcup_n d_n^{(m)}$$

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Then

$$\bigsqcup_{n\geq 0} d_{0,n} \sqsubseteq \bigsqcup_{n\geq 0} d_{1,n} \sqsubseteq \bigsqcup_{n\geq 0} d_{2,n} \sqsubseteq \ldots$$

and

$$\bigsqcup_{m\geq 0} d_{m,0} \sqsubseteq \bigsqcup_{m\geq 0} d_{m,1} \sqsubseteq \bigsqcup_{m\geq 0} d_{m,3} \sqsubseteq \ldots$$

Moreover

$$\bigsqcup_{m\geq 0}\left(\bigsqcup_{n\geq 0} d_{m,n}\right) = \bigsqcup_{k\geq 0} d_{k,k} = \bigsqcup_{n\geq 0}\left(\bigsqcup_{m\geq 0} d_{m,n}\right) .$$

$$
\dfrac{
\dfrac{\checkmark}{d_{k,k} \sqsubseteq \underbrace{\textstyle\bigsqcup_n d_{k,n}}}\,(\text{lub}\,1)
\qquad
\dfrac{\checkmark}{\underbrace{\textstyle\bigsqcup_n d_{k,n}} \sqsubseteq \bigsqcup_m \bigsqcup_n d_{m,n}}\,(\text{lub}\,1)
}{
\forall k \qquad d_{k,k} \sqsubseteq \bigsqcup_m \bigsqcup_n d_{m,n}
}\,(\text{trans})
$$

$$
\dfrac{\forall k \qquad d_{k,k} \sqsubseteq \bigsqcup_m \bigsqcup_n d_{m,n}}{\bigsqcup_k d_{k,k} \sqsubseteq \bigsqcup_m \bigsqcup_n d_{m,n}}\,(\text{lub}\,2)
$$

$$
\dfrac{\dfrac{\checkmark}{d_{max(m,n),max(m,n)}}\,(hyp) \qquad \dfrac{}{d_{max(m,n),max(m,n)}}\,(lub1)}{d_{m,n} \sqsubseteq \sim \qquad \sim \sqsubseteq \bigsqcup_k d_{k,k}}
$$

$$
\dfrac{\forall n \quad d_{m,n} \sqsubseteq \bigsqcup_k d_{k,k}}{\forall m \quad \bigsqcup_n d_{m,n} \sqsubseteq \bigsqcup_k d_{k,k}}\,(lub2)
$$

$$
\dfrac{\forall m \quad \bigsqcup_n d_{m,n} \sqsubseteq \bigsqcup_k d_{k,k}}{\bigsqcup_m \bigsqcup_n d_{m,n} \sqsubseteq \bigsqcup_k d_{k,k}}\,(lub2)
$$

(1) or (2) ~ which one?

Continuity and strictness

- If D and E are cpo's, the function f is continuous iff
  1. it is monotone, and
  2. it preserves lubs of chains, *i.e.* for all chains $d_0 \sqsubseteq d_1 \sqsubseteq \ldots$ in D, it is the case that

$$f(\bigsqcup_{n \geq 0} d_n) = \bigsqcup_{n \geq 0} f(d_n) \quad \text{in } E.$$

(1)

$$f(\bigsqcup_n d_n) \sqsubseteq \bigsqcup_n f(d_n)$$

(2)

$$\bigsqcup_n f(d_n) \sqsubseteq f(\bigsqcup_n d_n)$$