Fixed point property of $[\![\textbf{while}\ B\ \textbf{do}\ C]\!] \in (State \rightharpoonup State)$.

$$[\![\textbf{while}\ B\ \textbf{do}\ C]\!] = f_{[\![B]\!],[\![C]\!]}([\![\textbf{while}\ B\ \textbf{do}\ C]\!])$$

where, for each $b : State \to \{true, false\}$ and $c : State \rightharpoonup State$, we define

as $\boxed{f_{b,c} : (State \rightharpoonup State) \to (State \rightharpoonup State)}$

$$f_{b,c} = \lambda w \in (State \rightharpoonup State).\ \lambda s \in State.\ if\big(b(s), w(c(s)), s\big).$$

$$[\![\text{while } B \text{ do } C]\!] = \lambda s \in State.\ if\big([\![B]\!](s), [\![\text{while } B \text{ do } C]\!]([\![C]\!]s), s\big)$$

Fixed point property of $[\![\mathbf{while}\ B\ \mathbf{do}\ C]\!]$

$$[\![\mathbf{while}\ B\ \mathbf{do}\ C]\!] = f_{[\![B]\!],[\![C]\!]}([\![\mathbf{while}\ B\ \mathbf{do}\ C]\!])$$

where, for each $b : \mathit{State} \to \{\mathit{true}, \mathit{false}\}$ and $c : \mathit{State} \rightharpoonup \mathit{State}$, we define

$$f_{b,c} : (\mathit{State} \rightharpoonup \mathit{State}) \to (\mathit{State} \rightharpoonup \mathit{State})$$

as

$$f_{b,c} = \lambda w \in (\mathit{State} \rightharpoonup \mathit{State}).\ \lambda s \in \mathit{State}.\ \mathit{if}\big(b(s), w(c(s)), s\big).$$

- Why does $w = f_{[\![B]\!],[\![C]\!]}(w)$ have a solution?
- What if it has several solutions—which one do we take to be $[\![\mathbf{while}\ B\ \mathbf{do}\ C]\!]$? The one that makes $\rightsquigarrow$ operational sense!

Approximating $[\![\mathbf{while}\ B\ \mathbf{do}\ C]\!] \in (State \rightharpoonup State)$

$W_0, W_1, \ldots, W_n, \ldots$ approximating $[\![\text{while } B \text{ do } C]\!]$.

$W_0 = \perp \in (State \rightharpoonup State)$

totally undefined partial function.

$$W_{n+1} \stackrel{def}{=} f_{[\![B]\!],[\![C]\!]}(W_n)$$

$$= \lambda s \in State.\ if\,([\![B]\!]\,s, W_n([\![C]\!]\,s), s)$$

$$W_0 = \perp$$

$$W_1 = \lambda s.\ \mathit{if}\,(\llbracket B \rrbracket s,\ \perp(\llbracket C \rrbracket s),\ s)$$

$$= \lambda s. \begin{cases} \uparrow & \llbracket B \rrbracket s = tt \\ s & \text{otw} \end{cases}$$

$$W_2 = \lambda s.\ \mathit{if}\,(\llbracket B \rrbracket s,\ W_1(\llbracket C \rrbracket s),\ s)$$

$$= \lambda s. \begin{cases} s & \llbracket B \rrbracket s = ff \\ \uparrow & \llbracket B \rrbracket (\llbracket C \rrbracket s) = tt \\ \llbracket C \rrbracket s & \llbracket B \rrbracket (\llbracket C \rrbracket s) = ff \end{cases}$$

Approximating $[\![\mathbf{while}\ B\ \mathbf{do}\ C]\!]$

$$W_n = f_{[\![B]\!],[\![C]\!]}{}^n(\bot)$$

$$= \quad \lambda s \in State. \begin{cases} [\![C]\!]^k(s) & \text{if } \exists\, 0 \le k < n.\, [\![B]\!]([\![C]\!]^k(s)) = false \\ & \text{and } \forall\, 0 \le i < k.\, [\![B]\!]([\![C]\!]^i(s)) = true \\ \uparrow & \text{if } \forall\, 0 \le i < n.\, [\![B]\!]([\![C]\!]^i(s)) = true \end{cases}$$

$$W_0 \sqsubseteq W_1 \sqsubseteq \cdots \sqsubseteq W_n \sqsubseteq \cdots \qquad \sqsubseteq [\![\text{while } B \text{ do } C]\!]$$

approximations of ~ limit.

$$\bigcup_{n \in \mathbb{N}} W_n$$

$$[\![\text{while } B \text{ do } C]\!] = \bigcup_{n \in \mathbb{N}} f_{[\![B]\!],[\![C]\!]}^{\,n}(\bot)$$

$\Downarrow$

$[\![\text{while } B \text{ do } C]\!]$ is a fixed point of $f_{[\![B]\!],[\![C]\!]}$

$$D \stackrel{\text{def}}{=} (\mathit{State} \rightharpoonup \mathit{State})$$

- **Partial order $\sqsubseteq$ on D:**

$w \sqsubseteq w'$ iff for all $s \in \mathit{State}$, if w is defined at s then so is w' and moreover $w(s) = w'(s)$.

iff the graph of w is included in the graph of w'.

- **Least element $\perp \in D$ w.r.t. $\sqsubseteq$:**

$\perp$ = totally undefined partial function

= partial function with empty graph

(satisfies $\perp \sqsubseteq w$, for all $w \in D$).

If $w_0 \subseteq w_1 \subseteq \cdots \subseteq w_n \subseteq \cdots \qquad (n \in \mathbb{N})$

in $(State \to State)$

Then $\bigcup_{n \in \mathbb{N}} graph(w_n)$

is the graph of a function, say $w_\infty : State \to State$,

which we refer to as the limit of the (w_n).

$f_{[\![B]\!],[\![C]\!]}$ is monotone; that is $w \subseteq w'$

Then $f_{[\![B]\!],[\![C]\!]}(w) \subseteq f_{[\![B]\!],[\![C]\!]}(w')$

Topic 2

Least Fixed Points

Thesis

All domains of computation are
partial orders with a least element.

provide a notion
of approximation
(of information)

no information

Thesis

All domains of computation are partial orders with a least element.

All computable functions are monotonic.

$$x \sqsubseteq y \Longrightarrow f(x) \sqsubseteq f(y)$$

$$\sqsubseteq \ \subseteq D \times D = \{(d, d') \mid d, d' \in D\}$$

Partially ordered sets

A binary relation $\sqsubseteq$ on a set D is a partial order iff it is

reflexive: $\forall d \in D.\ d \sqsubseteq d$

transitive: $\forall d, d', d'' \in D.\ d \sqsubseteq d' \sqsubseteq d'' \Rightarrow d \sqsubseteq d''$

anti-symmetric: $\forall d, d' \in D.\ d \sqsubseteq d' \sqsubseteq d \Rightarrow d = d'$.

Such a pair $(D, \sqsubseteq)$ is called a partially ordered set, or poset.

$$\frac{}{x \sqsubseteq x}$$

$$\frac{x \sqsubseteq y \qquad y \sqsubseteq z}{x \sqsubseteq z}$$

$$\frac{x \sqsubseteq y \qquad y \sqsubseteq x}{x = y}$$

Domain of partial functions, $X \rightharpoonup Y$

Underlying set: all partial functions, f, with domain of definition $dom(f) \subseteq X$ and taking values in Y.

Partial order:

$$
\begin{array}{lll}
f \sqsubseteq g & \text{iff} & dom(f) \subseteq dom(g) \text{ and} \\
& & \forall x \in dom(f).\ f(x) = g(x) \\
& \text{iff} & graph(f) \subseteq graph(g)
\end{array}
$$

Monotonicity

$\sqsubseteq_D \quad \sqsubseteq_E$

- A function $f : D \rightarrow E$ between posets is monotone iff

$$\forall d, d' \in D.\ d \sqsubseteq_D d' \Rightarrow f(d) \sqsubseteq_E f(d').$$

$$\frac{x \sqsubseteq y}{f(x) \sqsubseteq f(y)} \quad (f \text{ monotone})$$

Least Elements

Suppose that D is a poset and that S is a subset of D.

An element $d \in S$ is the *least* element of S if it satisfies

$$\forall x \in S.\, d \sqsubseteq x \ .$$

say d_0 is a least element ① and ② d_1 is also a least element
then by ① $d_0 \sqsubseteq d_1$ and by ② $d_1 \sqsubseteq d_0$ so $d_0 = d_1$.

- Note that because $\sqsubseteq$ is anti-symmetric, S has at most one least element.
- Note also that a poset may not have least element.

Recall x is a fixed point of f if $f(x)=x$.

Pre-fixed points

Let D be a poset and $f : D \to D$ be ~~a function~~ monotone.

An element $d \in D$ is a pre-fixed point of f if it satisfies $f(d) \sqsubseteq d$.

The *least pre-fixed point* of f, if it exists, will be written

make computational sense..

$$\boxed{fix(f)}$$

It is thus (uniquely) specified by the two properties:

$$f(fix(f)) \sqsubseteq fix(f) \qquad \text{(lfp1)}$$

$$\forall d \in D.\ f(d) \sqsubseteq d \;\Rightarrow\; fix(f) \sqsubseteq d. \qquad \text{(lfp2)}$$

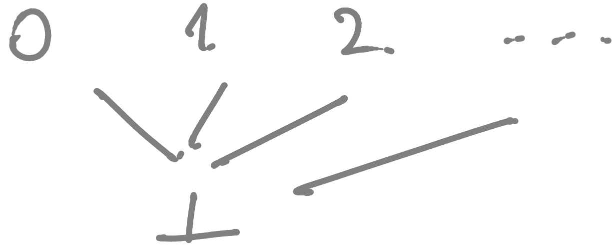
- A poset with no least element.

$(\mathbb{Z}, \leq)$ $\quad \cdots \leq -2 \leq -1 \leq 0 \leq 1 \leq 2 \leq \cdots$

$(\mathbb{N}, =)$ $\quad 0 \quad 1 \quad 2 \quad 3$

- Lifting: $(\mathbb{N}_\perp, \sqsubseteq)$ $\quad \mathbb{N}_\perp = \mathbb{N} \cup \{\perp\}$

$x \sqsubseteq y$ iff $(x = \perp)$ or $(x = y)$



$(\mathbb{N}_\perp, \sqsubseteq) \longrightarrow (\mathbb{N}_\perp, \sqsubseteq)$

a monotone function

with no least prefixed point ?

No !