Computable = λ-definable

Theorem. A partial function is computable if and only if it is λ-definable.

We already know that

$$\begin{aligned} & \text{Register Machine computable} \\ = \; & \text{Turing computable} \\ = \; & \text{partial recursive.} \end{aligned}$$

Using this, we break the theorem into two parts:

- every partial recursive function is λ-definable
- λ-definable functions are RM computable

Recall:

Representing primitive recursion

If $f \in \mathbb{N}^n \to \mathbb{N}$ is represented by a λ-term F and

$g \in \mathbb{N}^{n+2} \to \mathbb{N}$ is represented by a λ-term G,

we want to show λ-definability of the unique
$h \in \mathbb{N}^{n+1} \to \mathbb{N}$ satisfying $\boxed{h = \Phi_{f,g}(h)}$

where $\Phi_{f,g} \in (\mathbb{N}^{n+1} \to \mathbb{N}) \to (\mathbb{N}^{n+1} \to \mathbb{N})$ is given by

$$\begin{aligned}\Phi_{f,g}(h)(\vec{a}, a) \triangleq\ & \mathit{if}\ a = 0\ \mathit{then}\ f(\vec{a}) \\ & \mathit{else}\ g(\vec{a}, a-1, h(\vec{a}, a-1))\end{aligned}$$

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where $\Phi_{f,g} \in (\mathbb{N}^{n+1} \to \mathbb{N}) \to (\mathbb{N}^{n+1} \to \mathbb{N})$ is given by...

Strategy:

- show that $\Phi_{f,g}$ is λ-definable;

$\lambda z \vec{x} x.\, \mathit{If}(\mathit{Eq}_0\, x)(F\vec{x})(G\vec{x}(\mathit{Pred}\, x)(z\vec{x}(\mathit{Pred}\, x)))$

Representing primitive recursion

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Strategy:

- show that $\Phi_{f,g}$ is λ-definable;
- show that we can solve fixed point equations $\boxed{X = M\,X}$ up to β-conversion in the λ-calculus.

Curry's fixed point combinator $\mathbf{Y}$

$$\mathbf{Y} \triangleq \lambda f.\, (\lambda x.\, f(x\, x))(\lambda x.\, f(x\, x))$$

$$\boxed{\mathbf{Y}\, M =_\beta M(\mathbf{Y}\, M)}$$

Origins of Y

Naive set theory	λ calculus
Russell set: $R \triangleq \{x \mid \neg(x \in x)\}$	$R \triangleq \lambda x.\ \text{not}\,(x\ x)$
	$\boxed{\text{not} \triangleq \lambda b.\ \text{If } b \text{ False True}}$

Origins of Y

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Russell set : $R \triangleq \{x \mid \neg(x \in x)\}$	$R \triangleq \lambda x.\ \text{not}\,(x\,x)$
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$$Y\,\mathrm{not} =_\beta R\,R = (\lambda x.\ \mathrm{not}\,(x\,x))(\lambda x.\ \mathrm{not}\,(x\,x))$$

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Russell's Paradox: $R \in R \Leftrightarrow \neg(R \in R)$	$RR =_\beta \mathrm{not}(R\,R)$

$$Y\mathrm{not} =_\beta RR = (\lambda x.\, \mathrm{not}(x x))(\lambda x.\, \mathrm{not}(x x))$$

$$Y f = (\lambda x.\, f(x x))(\lambda x.\, f(x x))$$

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$$\mathbf{Y} \triangleq \lambda f.\,(\lambda x.\, f(x\,x))(\lambda x.\, f(x\,x))$$

satisfies $\mathbf{Y}\,M \quad \rightarrow \quad (\lambda x.\, M(x\,x))(\lambda x.\, M(x\,x))$

Curry's fixed point combinator $\mathbf{Y}$

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satisfies
$$\begin{aligned}\mathbf{Y}\,M &\to (\lambda x. M(x\,x))(\lambda x. M(x\,x))\\ &\to M((\lambda x. M(x\,x))(\lambda x. M(x\,x)))\end{aligned}$$

hence $\mathbf{Y}\,M \twoheadrightarrow M((\lambda x. M(x\,x))(\lambda x. M(x\,x))) \twoheadleftarrow M(\mathbf{Y}\,M)$.

So for all λ-terms M we have

$$\boxed{\mathbf{Y}\,M =_\beta M(\mathbf{Y}\,M)}$$

Turing's fixed point combinator

$$\Theta \triangleq A\,A$$

where $A \triangleq \lambda xy.\, y(xxy)$

Turing's fixed point combinator

$$\Theta \triangleq AA$$

where $A \triangleq \lambda xy.\, y(xxy)$

$$\Theta M = AAM = (\lambda xy.\, y(xxy))AM$$

Turing's fixed point combinator

$$\boxed{\begin{array}{lrcl} & \Theta & \triangleq & A\,A \\ \text{where} & A & \triangleq & \lambda xy.\,y(xxy) \end{array}}$$

$$\begin{aligned} \Theta M = A A M &= (\lambda xy.\,y(xxy))\,A\,M \\ &\twoheadrightarrow M(A A M) \\ &= M(\Theta M) \end{aligned}$$

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We now know that h can be represented by

$Y(\lambda z \vec{x} x.\,\mathbf{If}(\mathbf{Eq}_0\, x)(F\, \vec{x})(G\, \vec{x}\, (\mathbf{Pred}\, x)(z\, \vec{x}\, (\mathbf{Pred}\, x))))$.

Representing primitive recursion

Recall that the class **PRIM** of primitive recursive functions is the smallest collection of (total) functions containing the basic functions and closed under the operations of composition and primitive recursion.

Combining the results about λ-definability so far, we have: **every $f \in \mathbf{PRIM}$ is λ-definable**.

So for λ-definability of all recursive functions, we just have to consider how to represent minimization. Recall...

Minimization

Given a partial function $f \in \mathbb{N}^{n+1} \rightharpoonup \mathbb{N}$, define $\mu^n f \in \mathbb{N}^n \rightharpoonup \mathbb{N}$ by

$\mu^n f(\vec{x}) \triangleq$ least x such that $f(\vec{x}, x) = 0$ and for each $i = 0, \ldots, x-1$, $f(\vec{x}, i)$ is defined and > 0 (undefined if there is no such x)

So $\mu^n f(\vec{x}) = g(\vec{x}, 0)$ where in general $g(\vec{x}, x)$ satisfies

$$g(\vec{x}, x) = \text{if } f(\vec{x}, x) = 0 \text{ then } x \text{ else } g(\vec{x}, x+1)$$

Minimization

Given a partial function $f \in \mathbb{N}^{n+1} \rightharpoonup \mathbb{N}$, define
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$$\mu^n f(\vec{x}) \triangleq \begin{array}{l}\text{least } x \text{ such that } f(\vec{x}, x) = 0 \text{ and} \\ \text{for each } i = 0, \ldots, x-1,\ f(\vec{x}, i) \\ \text{is defined and } > 0 \\ (\text{undefined if there is no such } x)\end{array}$$

Can express $\mu^n f$ in terms of a fixed point equation:
$\mu^n f(\vec{x}) \equiv g(\vec{x}, 0)$ where g satisfies $\boxed{g = \Psi_f(g)}$
with $\Psi_f \in (\mathbb{N}^{n+1} \rightharpoonup \mathbb{N}) \to (\mathbb{N}^{n+1} \rightharpoonup \mathbb{N})$ defined by

$$\Psi_f(g)(\vec{x}, x) \equiv \mathit{if}\ f(\vec{x}, x) = 0\ \mathit{then}\ x\ \mathit{else}\ g(\vec{x}, x+1)$$

Representing minimization

Suppose $f \in \mathbb{N}^{n+1} \to \mathbb{N}$ (totally defined function) satisfies $\forall \vec{a}\, \exists a\, (f(\vec{a}, a) = 0)$, so that $\mu^n f \in \mathbb{N}^n \to \mathbb{N}$ is totally defined.

Thus for all $\vec{a} \in \mathbb{N}^n$, $\mu^n f(\vec{a}) = g(\vec{a}, 0)$ with $g = \Psi_f(g)$ and $\Psi_f(g)(\vec{a}, a)$ given by
$\mathit{if}\ (f(\vec{a}, a) = 0)\ \mathit{then}\ a\ \mathit{else}\ g(\vec{a}, a+1)$.

So if f is represented by a λ-term F, then $\mu^n f$ is represented by

$$\boxed{\lambda \vec{x}.\mathbf{Y}(\lambda z\, \vec{x}\, x.\, \mathbf{If}(\mathbf{Eq}_0(F\, \vec{x}\, x))\, x\, (z\, \vec{x}\, (\mathbf{Succ}\, x)))\, \vec{x}\, \underline{0}}$$

Recursive implies λ-definable

Fact: every partial recursive $f \in \mathbb{N}^n \rightharpoonup \mathbb{N}$ can be expressed in a standard form as $f = g \circ (\mu^n h)$ for some $g, h \in \mathbf{PRIM}$. (Follows from the proof that computable = partial-recursive.)

Hence every (total) recursive function is λ-definable.

More generally, every partial recursive function is λ-definable, but matching up $\uparrow$ with $\nexists\beta\mathbf{-nf}$ makes the representations more complicated than for total functions: see [Hindley, J.R. & Seldin, J.P. (CUP, 2008), chapter 4.]

Computable = λ-definable

Theorem. A partial function is computable if and only if it is λ-definable.

We already know that computable = partial recursive $\Rightarrow$ λ-definable. So it just remains to see that **λ-definable functions are RM computable**. To show this one can

- code λ-terms as numbers (ensuring that operations for constructing and deconstructing terms are given by RM computable functions on codes)
- write a RM interpreter for (normal order) β-reduction.

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Numerical coding of λ-terms

Fix an enumeration $x_0, x_1, x_2, \ldots$ of the set of variables.

For each λ-term M, define $\ulcorner M \urcorner \in \mathbb{N}$ by

$$\boxed{\begin{aligned}
\ulcorner x_i \urcorner &= \ulcorner [0, i] \urcorner \\
\ulcorner \lambda x_i . M \urcorner &= \ulcorner [1, i, \ulcorner M \urcorner] \urcorner \\
\ulcorner M N \urcorner &= \ulcorner [2, \ulcorner M \urcorner, \ulcorner N \urcorner] \urcorner
\end{aligned}}$$

(where $\ulcorner [n_0, n_1, \ldots, n_k] \urcorner$ is the numerical coding of lists of numbers from p43).

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The details are straightforward, if tedious.

Recall:

$$\frac{M_1 =_\alpha M_1' \quad M_1' \to_{NOR} M_2' \quad M_2' =_\alpha M_2}{M_1 \to_{NOR} M_2}$$

$$\frac{M \to_{NOR} M'}{\lambda x. M \to_{NOR} \lambda x. M'}$$

$$\frac{M_1 \to_{NOR} M_1'}{M_1 M_2 \to_{NOR} M_1' M_2}$$

$$\frac{}{(\lambda x. M)\, M' \to_{NOR} M[M'/x]}$$

$$\frac{M \to_{NOR} M'}{U M \to_{NOR} U M'}$$

where
$$\begin{cases} U ::= x \mid U N \\ N ::= \lambda x. N \mid U \end{cases}$$

β-normal forms

"neutral" forms

Summary

- Formalization of intuitive notion of ALGORITHM in several equivalent ways

  cf. "Church-Turing Thesis"

- Limitative results: $\begin{cases} \text{undecidable problems} \\ \text{uncomputable functions} \end{cases}$

"programs as data" + diagonalization