

Type Systems

Lecture 9: Classical Logic

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Where We Are

We have seen the Curry Howard correspondence:

- **Intuitionistic** propositional logic \longleftrightarrow Simply-typed lambda calculus
- Second-order **intuitionistic** logic \longleftrightarrow Polymorphic lambda calculus

We have seen effectful programs:

- State
- I/O
- Monads

But what about:

- Control operators (eg, exceptions, **goto**, etc)
- **Classical** logic

A Review of Intuitionistic Propositional Logic

$$\frac{P \in \Psi}{\Psi \vdash P \text{ true}} \text{ HYP}$$

$$\frac{}{\Psi \vdash \top \text{ true}} \text{ T1}$$

$$\frac{\Psi \vdash P \text{ true} \quad \Psi \vdash Q \text{ true}}{\Psi \vdash P \wedge Q \text{ true}} \wedge \text{I}$$

$$\frac{\Psi \vdash P_1 \wedge P_2 \text{ true}}{\Psi \vdash P_i \text{ true}} \wedge \text{E}_i$$

$$\frac{\Psi, P \vdash Q \text{ true}}{\Psi \vdash P \supset Q \text{ true}} \supset \text{I}$$

$$\frac{\Psi \vdash P \supset Q \text{ true} \quad \Psi \vdash P \text{ true}}{\Psi \vdash Q \text{ true}} \supset \text{E}$$

Disjunction and Falsehood

$$\frac{\Psi \vdash P \text{ true}}{\Psi \vdash P \vee Q \text{ true}} \vee I_1$$

$$\frac{\Psi \vdash Q \text{ true}}{\Psi \vdash P \vee Q \text{ true}} \vee I_2$$

$$\frac{\Psi \vdash P \vee Q \text{ true} \quad \Psi, P \vdash R \text{ true} \quad \Psi, Q \vdash R \text{ true}}{\Psi \vdash R \text{ true}} \vee E$$

(no intro for \perp)

$$\frac{\Psi \vdash \perp \text{ true}}{\Psi \vdash R \text{ true}} \perp E$$

Intuitionistic Propositional Logic

- Key judgement: $\Psi \vdash R$ true
 - “If everything in Ψ is true, then R is true”
- Negation $\neg P$ is a derived notion
 - Definition: $\neg P = P \rightarrow \perp$
 - “Not P ” means “ P implies false”
 - To *refute* P means to give a *proof* that P implies false

What if we treat refutations as a first-class notion?

A Calculus of Truth and Falsehood

Propositions $A ::= \top \mid A \wedge B \mid \perp \mid A \vee B \mid \neg A$

True contexts $\Gamma ::= \cdot \mid \Gamma, A$

False contexts $\Delta ::= \cdot \mid \Delta, A$

Proofs $\Gamma; \Delta \vdash A$ true If Γ is true and Δ is false, A is true

Refutations $\Gamma; \Delta \vdash A$ false If Γ is true and Δ is false, A is false

Contradictions $\Gamma; \Delta \vdash \text{contr}$ Γ and Δ contradict one another

- $\neg A$ is primitive (no implication $A \rightarrow B$)
- Eventually, we'll encode it as $\neg A \vee B$

$$\frac{A \in \Gamma}{\Gamma; \Delta \vdash A \text{ true}} \text{HYP}$$

(No rule for \perp) $\frac{}{\Gamma; \Delta \vdash \top \text{ true}} \top P$

$$\frac{\Gamma; \Delta \vdash A \text{ true} \quad \Gamma; \Delta \vdash B \text{ true}}{\Gamma; \Delta \vdash A \wedge B \text{ true}} \wedge P$$

$$\frac{\Gamma; \Delta \vdash A \text{ true}}{\Gamma; \Delta \vdash A \vee B \text{ true}} \vee P_1$$

$$\frac{\Gamma; \Delta \vdash B \text{ true}}{\Gamma; \Delta \vdash A \vee B \text{ true}} \vee P_2$$

$$\frac{\Gamma; \Delta \vdash A \text{ false}}{\Gamma; \Delta \vdash \neg A \text{ true}} \neg P$$

Refutations

$$\frac{A \in \Delta}{\Gamma; \Delta \vdash A \text{ false}} \text{HYP}$$

(No rule for \top) $\frac{}{\Gamma; \Delta \vdash \perp \text{ false}} \perp R$

$$\frac{\Gamma; \Delta \vdash A \text{ false} \quad \Gamma; \Delta \vdash B \text{ false}}{\Gamma; \Delta \vdash A \vee B \text{ false}} \vee R$$

$$\frac{\Gamma; \Delta \vdash A \text{ false}}{\Gamma; \Delta \vdash A \wedge B \text{ false}} \wedge R_1$$

$$\frac{\Gamma; \Delta \vdash B \text{ false}}{\Gamma; \Delta \vdash A \wedge B \text{ false}} \wedge R_2$$

$$\frac{\Gamma; \Delta \vdash A \text{ true}}{\Gamma; \Delta \vdash \neg A \text{ false}} \neg R$$

75% of the Way to Classical Logic

Connective	To Prove	To Refute
\top	Do nothing	Impossible!
$A \wedge B$	Prove <i>A and</i> prove <i>B</i>	Refute <i>A or</i> refute <i>B</i>
\perp	Impossible!	Do nothing
$A \vee B$	Prove <i>A or</i> prove <i>B</i>	Refute <i>A and</i> refute <i>B</i>
$\neg A$	Refute <i>A</i>	Prove <i>A</i>

Something We Can Prove: A entails $\neg\neg A$

$$\frac{\frac{\frac{}{A; \cdot \vdash A \text{ true}}{\text{HYP}}}{A; \cdot \vdash \neg A \text{ false}}{\neg R}}{A; \cdot \vdash \neg\neg A \text{ true}}{\neg P}$$

Something We Cannot Prove: $\neg\neg A$ entails A

$$\frac{???}{\neg\neg A; \cdot \vdash A \text{ true}}$$

- There is no rule that applies in this case
- Proofs and refutations are mutually recursive
- But we have no way to use assumptions!

Something Else We Cannot Prove: $A \wedge B$ entails A

$$\frac{???}{A \wedge B; \cdot \vdash A \text{ true}}$$

- This is intuitionistically valid: $\lambda x : A \times B. \text{fst } x$
- But it's not derivable here
- Again, we can't use hypotheses nontrivially

A Bold Assumption

- Proofs and refutations are perfectly symmetrical
- This suggests the following idea:
 1. To refute A means to give direct evidence it is false
 2. This is also how we prove $\neg A$
 3. If we show a contradiction from assuming A is false, we have proved it
 4. If we can show a contradiction from assuming A is true, we have refuted it

$$\frac{\Gamma; \Delta, A \vdash \text{contr}}{\Gamma; \Delta \vdash A \text{ true}}$$

$$\frac{\Gamma, A; \Delta \vdash \text{contr}}{\Gamma; \Delta \vdash A \text{ false}}$$

Contradictions

$$\frac{\Gamma; \Delta \vdash A \text{ true} \quad \Gamma; \Delta \vdash A \text{ false}}{\Gamma; \Delta \vdash \text{contr}} \text{ CONTR}$$

- A contradiction arises when A has a proof *and* a refutation

Double Negation Elimination

	$\overline{\neg\neg A; A \vdash A \text{ false}}$
	$\overline{\neg\neg A; A \vdash \neg A \text{ true}}$
$\overline{\neg\neg A; A \vdash \neg\neg A \text{ true}}$	$\overline{\neg\neg A; A \vdash \neg\neg A \text{ false}}$
$\overline{\neg\neg A; A \vdash \text{contr}}$	
$\overline{\neg\neg A; \cdot \vdash A \text{ true}}$	

Projections: $A \wedge B$ entails A

$$\frac{\frac{}{A \wedge B; A \vdash A \wedge B \text{ true}} \quad \frac{\frac{}{A \wedge B; A \vdash A \text{ false}}}{A \wedge B; A \vdash A \wedge B \text{ false}}}{A \wedge B; A \vdash \text{contr}}}{A \wedge B; \cdot \vdash A \text{ true}}$$

Projections: $A \vee B$ false entails A false

$$\frac{\frac{\frac{}{A; A \vee B \vdash A \vee B \text{ false}}{} \quad \frac{\frac{}{A; A \vee B \vdash A \text{ true}}{} \quad \frac{}{A; A \vee B \vdash A \vee B \text{ true}}{A; A \vee B \vdash A \vee B \text{ true}}{A; A \vee B \vdash \text{contr}}}{A; A \vee B \vdash \text{contr}}}{\therefore A \vee B \vdash A \text{ false}}$$

The Excluded Middle

$$\begin{array}{c} \vdots \\ \hline \cdot; A \vee \neg A \vdash A \text{ false} \\ \hline \cdot; A \vee \neg A \vdash \neg A \text{ true} \\ \hline \cdot; A \vee \neg A \vdash A \vee \neg A \text{ true} \qquad \cdot; A \vee \neg A \vdash A \vee \neg A \text{ false} \\ \hline \cdot; A \vee \neg A \vdash \text{contr} \\ \hline \cdot \vdash A \vee \neg A \text{ true} \end{array}$$

Proof (and Refutation) Terms

Propositions	A	$::=$	$\top \mid A \wedge B \mid \perp \mid A \vee B \mid \neg A$
True contexts	Γ	$::=$	$\cdot \mid \Gamma, x : A$
False contexts	Δ	$::=$	$\cdot \mid \Delta, u : A$
Values	e	$::=$	$\langle \rangle \mid \langle e, e' \rangle \mid L e \mid R e \mid \text{not}(k)$ $\mid \mu u : A. c$
Continuations	k	$::=$	$[] \mid [k, k'] \mid \text{fst } k \mid \text{snd } k \mid \text{not}(e)$ $\mid \mu x : A. c$
Contradictions	c	$::=$	$\langle e \mid_A k \rangle$

Expressions — Proof Terms

$$\frac{x : A \in \Gamma}{\Gamma; \Delta \vdash x : A \text{ true}} \text{HYP}$$

(No rule for \perp)

$$\frac{}{\Gamma; \Delta \vdash \langle \rangle : \top \text{ true}} \text{TP}$$

$$\frac{\Gamma; \Delta \vdash e : A \text{ true} \quad \Gamma; \Delta \vdash e' : B \text{ true}}{\Gamma; \Delta \vdash \langle e, e' \rangle : A \wedge B \text{ true}} \wedge P$$

$$\frac{\Gamma; \Delta \vdash e : A \text{ true}}{\Gamma; \Delta \vdash L e : A \vee B \text{ true}} \vee P_1$$

$$\frac{\Gamma; \Delta \vdash e : B \text{ true}}{\Gamma; \Delta \vdash R e : A \vee B \text{ true}} \vee P_2$$

$$\frac{\Gamma; \Delta \vdash k : A \text{ false}}{\Gamma; \Delta \vdash \text{not}(k) : \neg A \text{ true}} \neg P$$

Continuations — Refutation Terms

$$\frac{x : A \in \Delta}{\Gamma; \Delta \vdash x : A \text{ false}} \text{HYP}$$

(No rule for \top)

$$\frac{}{\Gamma; \Delta \vdash [] : \perp \text{ false}} \perp R$$

$$\frac{\Gamma; \Delta \vdash k : A \text{ false} \quad \Gamma; \Delta \vdash k' : B \text{ false}}{\Gamma; \Delta \vdash [k, k'] : A \vee B \text{ false}} \vee R$$

$$\frac{\Gamma; \Delta \vdash k : A \text{ false}}{\Gamma; \Delta \vdash \text{fst } k : A \wedge B \text{ false}} \wedge R_1$$

$$\frac{\Gamma; \Delta \vdash k : B \text{ false}}{\Gamma; \Delta \vdash \text{snd } k : A \wedge B \text{ false}} \wedge R_2$$

$$\frac{\Gamma; \Delta \vdash e : A \text{ true}}{\Gamma; \Delta \vdash \text{not}(e) : \neg A \text{ false}} \neg R$$

Contradictions

$$\frac{\Gamma; \Delta \vdash e : A \text{ true} \quad \Gamma; \Delta \vdash k : A \text{ false}}{\Gamma; \Delta \vdash \langle e \mid_A k \rangle \text{ contr}} \text{ CONTR}$$

$$\frac{\Gamma; \Delta, u : A \vdash c \text{ contr}}{\Gamma; \Delta \vdash \mu u : A. c : A \text{ true}}$$

$$\frac{\Gamma, x : A; \Delta \vdash c \text{ contr}}{\Gamma; \Delta \vdash \mu x : A. c : A \text{ false}}$$

Operational Semantics

$$\langle \langle e_1, e_2 \rangle \mid_{A \wedge B} \text{fst } k \rangle \mapsto \langle e_1 \mid_A k \rangle$$

$$\langle \langle e_1, e_2 \rangle \mid_{A \wedge B} \text{snd } k \rangle \mapsto \langle e_2 \mid_B k \rangle$$

$$\langle L e \mid_{A \vee B} [k_1, k_2] \rangle \mapsto \langle e \mid_A k_1 \rangle$$

$$\langle R e \mid_{A \vee B} [k_1, k_2] \rangle \mapsto \langle e \mid_B k_2 \rangle$$

$$\langle \text{not}(k) \mid_{\neg A} \text{not}(e) \rangle \mapsto \langle e \mid_A k \rangle$$

$$\langle \mu u : A. c \mid_A k \rangle \mapsto [k/u]c$$

$$\langle e \mid_A \mu x : A. c \rangle \mapsto [e/x]c$$

A Bit of Non-Determinism

$$\langle \mu u : A. c \mid_A \mu x : A. c' \rangle \mapsto ?$$

- Two rules apply!
- Different choices of priority correspond to *evaluation order*
- Similar situation in the simply-typed lambda calculus
- The STLC is *confluent*, so evaluation order doesn't matter
- But in the classical case, evaluation order matters a lot!

Metatheory: Substitution

- If $\Gamma; \Delta \vdash e : A$ true then
 1. If $\Gamma, x : A; \Delta \vdash e' : C$ true then $\Gamma; \Delta \vdash [e/x]e' : C$ true.
 2. If $\Gamma, x : A; \Delta \vdash k : C$ false then $\Gamma; \Delta \vdash [e/x]k : C$ false.
 3. If $\Gamma, x : A; \Delta \vdash c$ contr then $\Gamma; \Delta \vdash [e/x]c$ contr.
- If $\Gamma; \Delta \vdash k : A$ false then
 1. If $\Gamma; \Delta, u : A \vdash e' : C$ true then $\Gamma; \Delta \vdash [k/u]e' : C$ true.
 2. If $\Gamma; \Delta, x : A \vdash k' : C$ false then $\Gamma; \Delta \vdash [k/u]k' : C$ false.
 3. If $\Gamma; \Delta, u : A \vdash c$ contr then $\Gamma; \Delta \vdash [k/u]c$ contr.
- We *also* need to prove weakening and exchange!
- Because there are 2 kinds of assumptions, and 3 kinds of judgement, there are $2 \times 3 = 6$ lemmas!

What Is This For?

- We have introduced a proof theory for classical logic
- Expected tautologies and metatheory holds...
- ...but it looks totally different from STLC?
- Computationally, this is a calculus for *stack machines*
- Related to *continuation passing style* (next lecture!)

Questions

1. Show that $\neg A \vee B, A; \cdot \vdash B$ true is derivable
2. Show that $\neg(\neg A \wedge \neg B); \cdot \vdash A \vee B$ true is derivable
3. Prove substitution for values (you may assume exchange and weakening hold).