Type Systems

Lecture 10: Classical Logic and Continuation-Passing Style

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Proof (and Refutation) Terms

$$
\begin{array}{llll}
\text{Propositions} & A & ::= & \top \mid A \wedge B \mid \bot \mid A \vee B \mid \neg A \\
\text{True contexts} & \Gamma & ::= & \cdot \mid \Gamma, x : A \\
\text{False contexts} & \Delta & ::= & \cdot \mid \Delta, u : A \\
\text{Values} & e & ::= & \langle\rangle \mid \langle e, e' \rangle \mid \mathsf{L}\, e \mid \mathsf{R}\, e \mid \mathsf{not}(k) \\
 & & \mid & \mu u : A.\, c \\
\text{Continuations} & k & ::= & [] \mid [k, k'] \mid \mathsf{fst}\, k \mid \mathsf{snd}\, k \mid \mathsf{not}(e) \\
 & & \mid & \mu x : A.\, c \\
\text{Contradictions} & c & ::= & \langle e \mid_A k \rangle
\end{array}
$$

Expressions — Proof Terms

$$\frac{x : A \in \Gamma}{\Gamma; \Delta \vdash x : A \text{ true}}\ \textsc{Hyp}$$

(No rule for $\bot$)

$$\frac{}{\Gamma; \Delta \vdash \langle\rangle : \top \text{ true}}\ \top\mathrm{P}$$

$$\frac{\Gamma; \Delta \vdash e : A \text{ true} \qquad \Gamma; \Delta \vdash e' : B \text{ true}}{\Gamma; \Delta \vdash \left\langle e, e' \right\rangle : A \wedge B \text{ true}}\ \wedge\mathrm{P}$$

$$\frac{\Gamma; \Delta \vdash e : A \text{ true}}{\Gamma; \Delta \vdash \mathrm{L}\, e : A \vee B \text{ true}}\ \vee\mathrm{P}_1 \qquad \frac{\Gamma; \Delta \vdash e : B \text{ true}}{\Gamma; \Delta \vdash \mathrm{R}\, e : A \vee B \text{ true}}\ \vee\mathrm{P}_2$$

$$\frac{\Gamma; \Delta \vdash k : A \text{ false}}{\Gamma; \Delta \vdash \mathrm{not}(k) : \neg A \text{ true}}\ \neg\mathrm{P}$$

Continuations — Refutation Terms

$$\frac{x : A \in \Delta}{\Gamma; \Delta \vdash x : A \text{ false}}\ \textsc{Hyp}$$

(No rule for $\top$)

$$\frac{}{\Gamma; \Delta \vdash [] : \bot \text{ false}}\ \bot R$$

$$\frac{\Gamma; \Delta \vdash k : A \text{ false} \qquad \Gamma; \Delta \vdash k' : B \text{ false}}{\Gamma; \Delta \vdash \left[k, k'\right] : A \vee B \text{ false}}\ \vee R$$

$$\frac{\Gamma; \Delta \vdash k : A \text{ false}}{\Gamma; \Delta \vdash \text{fst}\, k : A \wedge B \text{ false}}\ \wedge R_1 \qquad \frac{\Gamma; \Delta \vdash k : B \text{ false}}{\Gamma; \Delta \vdash \text{snd}\, k : A \wedge B \text{ false}}\ \wedge R_2$$

$$\frac{\Gamma; \Delta \vdash e : A \text{ true}}{\Gamma; \Delta \vdash \text{not}(e) : \neg A \text{ false}}\ \neg R$$

Contradictions

$$\frac{\Gamma; \Delta \vdash e : A \text{ true} \qquad \Gamma; \Delta \vdash k : A \text{ false}}{\Gamma; \Delta \vdash \langle e \mid_A k \rangle \text{ contr}} \textsc{Contr}$$

$$\frac{\Gamma; \Delta, u : A \vdash c \text{ contr}}{\Gamma; \Delta \vdash \mu u : A.\, c : A \text{ true}} \qquad \frac{\Gamma, x : A; \Delta \vdash c \text{ contr}}{\Gamma; \Delta \vdash \mu x : A.\, c : A \text{ false}}$$

Operational Semantics

$$
\begin{array}{lcl}
\langle \langle e_1, e_2 \rangle \mid_{A \wedge B} \mathsf{fst}\, k \rangle & \mapsto & \langle e_1 \mid_A k \rangle \\
\langle \langle e_1, e_2 \rangle \mid_{A \wedge B} \mathsf{snd}\, k \rangle & \mapsto & \langle e_2 \mid_B k \rangle \\
\langle \mathsf{L}\, e \mid_{A \vee B} [k_1, k_2] \rangle & \mapsto & \langle e \mid_A k_1 \rangle \\
\langle \mathsf{R}\, e \mid_{A \vee B} [k_1, k_2] \rangle & \mapsto & \langle e \mid_B k_2 \rangle \\
\langle \mathsf{not}(k) \mid_{\neg A} \mathsf{not}(e) \rangle & \mapsto & \langle e \mid_A k \rangle \\
\hline
\langle \mu u : A.\, c \mid_A k \rangle & \mapsto & [k/u]c \\
\langle e \mid_A \mu x : A.\, c \rangle & \mapsto & [e/x]c
\end{array}
$$

Type Safety?

Preservation If $\cdot; \cdot \vdash c$ contr and $c \rightsquigarrow c'$ then $\cdot; \cdot \vdash c'$ contr.

Proof By *case analysis* on evaluation derivations!

(We don't even need induction!)

Type Preservation

$\langle\langle e_1, e_2\rangle \mid_{A\wedge B} \mathsf{fst}\, k\rangle \rightsquigarrow \langle e_1 \mid_A k\rangle$ Assumption

$$\frac{\overbrace{\Gamma;\Delta \vdash \langle e_1, e_2\rangle : A \wedge B \text{ true}}^{(1)} \qquad \overbrace{\Gamma;\Delta \vdash \mathsf{fst}\, k : A \wedge B \text{ false}}^{(2)}}{\Gamma;\Delta \vdash \langle\langle e_1, e_2\rangle \mid_{A\wedge B} \mathsf{fst}\, k\rangle \text{ contr}}$$ Assumption

$$\frac{\overbrace{\Gamma;\Delta \vdash e_1 : A \text{ true}}^{(3)} \qquad \Gamma;\Delta \vdash e_2 : B \text{ true}}{\Gamma;\Delta \vdash \langle e_1, e_2\rangle : A \wedge B \text{ true}} \wedge P$$ Analysis of (1)

$$\frac{\overbrace{\Gamma;\Delta \vdash k : A \text{ false}}^{(4)}}{\Gamma;\Delta \vdash \mathsf{fst}\, k : A \wedge B \text{ false}} \wedge R_1$$ Analysis of (2)

$\cdot;\cdot \vdash \langle e_1 \mid_A k\rangle \text{ contr}$ By rule on (3), (

Progress?

Progress? If $\cdot; \cdot \vdash c$ contr then $c \leadsto c'$ (or c final).

Proof:

1. A closed term c is a contradiction
2. Hopefully, there aren't any contradictions!
3. So this theorem is vacuous (assuming classical logic is consistent)

Making Progress Less Vacuous

$$
\begin{array}{llll}
\text{Propositions} & A & ::= & \ldots \mid \mathsf{ans} \\
\text{Values} & e & ::= & \ldots \mid \mathsf{halt} \\
\text{Continuations} & k & ::= & \ldots \mid \mathsf{done}
\end{array}
$$

$$
\frac{}{\Gamma; \Delta \vdash \mathsf{halt} : \mathsf{ans}\ \mathsf{true}} \qquad \frac{}{\Gamma; \Delta \vdash \mathsf{done} : \mathsf{ans}\ \mathsf{false}}
$$

Progress If $\cdot; \cdot \vdash c$ contr then $c \rightsquigarrow c'$ or $c = \langle \text{halt} \mid_{\text{ans}} \text{done} \rangle$.

Proof By induction on typing derivations

The Price of Progress

$$
\dfrac{
\dfrac{
\dfrac{\Gamma; \Delta, A \vdash \mathsf{ans}\ \ \mathsf{true} \qquad \Gamma; \Delta, A \vdash \mathsf{ans}\ \mathsf{false}}{\Gamma; \Delta, A \vdash \mathsf{contr}}
}{\Gamma; \Delta \vdash A\ \ \mathsf{true}}
\qquad
\dfrac{
\dfrac{
\dfrac{\Gamma, A; \Delta \vdash \mathsf{ans}\ \ \mathsf{true} \qquad \Gamma, A; \Delta \vdash \mathsf{ans}\ \mathsf{false}}{\Gamma, A; \Delta \vdash \mathsf{contr}}
}{\Gamma; \Delta \vdash A\ \mathsf{false}}
}{\Gamma; \Delta \vdash \neg A\ \ \mathsf{true}}
}{\Gamma; \Delta \vdash A \wedge \neg A\ \ \mathsf{true}}
$$

- As a term:

$$\langle \mu u : A.\, \langle \mathsf{halt} \mid \mathsf{done} \rangle, \mathsf{not}(\mu x : A.\, \langle \mathsf{halt} \mid \mathsf{done} \rangle) \rangle$$

- Adding a halt configuration makes classical logic inconsistent – $A \wedge \neg A$ is derivable

Embedding Classical Logic into Intuitionistic Logic

- Intuitionistic logic has a clean computational reading
- Classical logic *almost* has a clean computational reading
- Q: Is there any way to equip classical logic with computational meaning?
- A: Embed classical logic *into* intuitionistic logic

The Double Negation Translation

- Fix an intuitionistic proposition p
- Define "quasi-negation" $\sim X$ as $X \to p$
- Now, we can define a translation on types as follows:

$$
\begin{array}{lcl}
(\neg A)^\circ & = & \sim A^\circ \\
\top^\circ & = & 1 \\
(A \wedge B)^\circ & = & A^\circ \times B^\circ \\
\perp^\circ & = & p \\
(A \vee B)^\circ & = & \sim\sim(A^\circ + B^\circ)
\end{array}
$$

Triple-Negation Elimination

In general, $\neg\neg X \to X$ is not derivable constructively. However, the following *is* derivable:

Lemma For all X, there is a function tne : $(\sim\sim\sim X) \to \sim X$

$$
\dfrac{\dots \qquad \dfrac{\dfrac{\dfrac{\dots \vdash q : X \to p \qquad \dots \vdash x : X}{k : \sim\sim\sim X, x : X, q : \sim X \vdash q\,x : p}}{k : \sim\sim\sim X, x : X \vdash \lambda q.\, q\,x : \sim\sim X}}{}}{\dfrac{\dfrac{k : \sim\sim\sim X, x : X \vdash k\,(\lambda q.\, q\,x) : p}{k : \sim\sim\sim X \vdash \lambda x.\, k\,(\lambda q.\, q\,a) : \sim X}}{\cdot \vdash \underbrace{\lambda k.\, \lambda a.\, k\,(\lambda q.\, q\,a)}_{\text{tne}} : (\sim\sim\sim X) \to \sim X}}
$$

Intuitionistic Double Negation Elimination

Lemma For all A, there is a term dne_A such that

$$\cdot \vdash \mathrm{dne}_A : {\sim}{\sim}A^\circ \to A^\circ$$

Proof By induction on A.

$$\begin{array}{lcl}
\mathrm{dne}_\top & = & \lambda x.\, \langle\rangle \\
\mathrm{dne}_{A\wedge B} & = & \lambda p.\, \left\langle \begin{array}{l} \mathrm{dne}_A\,(\lambda k.\, q\,(\lambda p.\, k\,(\mathrm{fst}\,p))), \\ \mathrm{dne}_B\,(\lambda k.\, q\,(\lambda p.\, k\,(\mathrm{snd}\,p))) \end{array} \right\rangle \\
\mathrm{dne}_\bot & = & \lambda q.\, q\,(\lambda x.\, x) \\
\mathrm{dne}_{A\vee B} & = & \lambda q : {\sim}{\sim}\underbrace{{\sim}{\sim}(A^\circ \vee B^\circ)}_{(A\vee B)^\circ}.\, \mathrm{tne}\,q \\
\mathrm{dne}_{\neg A} & = & \lambda q : {\sim}{\sim}\underbrace{({\sim}A^\circ)}_{(\neg A)^\circ}.\, \mathrm{tne}\,q
\end{array}$$

Double Negation Elimination for $\perp$

$$\dfrac{\dfrac{}{q : (p \to p) \to p \vdash q : (p \to p) \to p} \qquad \dfrac{\dfrac{}{q : (p \to p) \to p, x : p \vdash x : p}}{q : (p \to p) \to p \vdash \lambda x : p.\, x : p}}{\dfrac{q : (p \to p) \to p \vdash q\,(\lambda x : p.\, x) : p}{\dfrac{\cdot \vdash \lambda q : (p \to p) \to p.\, q\,(\lambda x : p.\, x) : ((p \to p) \to p) \to p}{\dfrac{\cdot \vdash \lambda q : \sim\sim p.\, q\,(\lambda x : p.\, x) : \sim\sim p \to p}{\cdot \vdash \lambda q : \sim\sim\perp^\circ.\, q\,(\lambda x : p.\, x) : \sim\sim\perp^\circ \to \perp^\circ}}}}$$

Translating Derivations

Theorem Classical terms embed into intutionistic terms:

1. If $\Gamma; \Delta \vdash e : A$ true then $\Gamma^\circ, \sim\Delta \vdash e^\circ : A^\circ$.
2. If $\Gamma; \Delta \vdash k : A$ false then $\Gamma^\circ, \sim\Delta \vdash k^\circ : \sim A^\circ$.
3. If $\Gamma; \Delta \vdash c$ contr then $\Gamma^\circ, \sim\Delta \vdash c^\circ : p$.

Proof By induction on derivations – but first, we have to define the translation!

Translating Contexts

Translating Value Contexts:

$$\begin{array}{lcl}(\cdot)^\circ & = & \cdot \\ (\Gamma, x : A)^\circ & = & \Gamma^\circ, x : A^\circ\end{array}$$

Translating Continuation Contexts:

$$\begin{array}{lcl}\sim(\cdot) & = & \cdot \\ \sim(\Gamma, x : A) & = & \sim\Gamma, x : \sim A^\circ\end{array}$$

Translating Contradictions

$$\frac{\Gamma; \Delta \vdash e : A \text{ true} \qquad \Gamma; \Delta \vdash k : A \text{ false}}{\Gamma; \Delta \vdash \langle e \mid_A k \rangle \text{ contr}}\ \textsc{Contr}$$

Define:

$$\langle e \mid_A k \rangle \quad = \quad k^\circ\, e^\circ$$

Translating (Most) Expressions

$$
\begin{array}{lcl}
x^\circ & = & x \\
\langle\rangle^\circ & = & \langle\rangle \\
\langle e_1, e_2\rangle^\circ & = & \langle {e_1}^\circ, {e_2}^\circ\rangle \\
(\mathsf{L}\, e)^\circ & = & \lambda k : {\sim}(A^\circ + B^\circ).\, k\,(\mathsf{L}\, e^\circ) \\
(\mathsf{R}\, e)^\circ & = & \lambda k : {\sim}(A^\circ + B^\circ).\, k\,(\mathsf{R}\, e^\circ) \\
(\mathsf{not}(k))^\circ & = & k^\circ
\end{array}
$$

Translating (Most) Continuations

$$
\begin{array}{lcl}
x^\circ & = & x \\
[]^\circ & = & \lambda x : \mathsf{ans}.\, x \\
[k_1, k_2]^\circ & = & \lambda k : {\sim}{\sim}(A^\circ + B^\circ). \\
 & & \quad k\ (\lambda i : A^\circ + B^\circ. \\
 & & \qquad \mathsf{case}(i, \mathsf{L}\, x \to k_1^\circ\, x, \mathsf{R}\, y \to k_2^\circ\, y)) \\
(\mathsf{fst}\, k)^\circ & = & \lambda p : (A^\circ \times B^\circ).\, k^\circ\, (\mathsf{fst}\, p) \\
(\mathsf{snd}\, k)^\circ & = & \lambda p : (A^\circ \times B^\circ).\, k^\circ\, (\mathsf{snd}\, p) \\
(\mathsf{not}(e))^\circ & = & \lambda k : \underbrace{{\sim}A^\circ}_{(\neg A)^\circ}.\, k\, e^\circ
\end{array}
$$

Translating Proof by Contradiction

$$\frac{\Gamma; \Delta, u : A \vdash c \text{ contr}}{\Gamma; \Delta \vdash \mu u : A.\, c : A \text{ true}}$$

1	$\Gamma^\circ, {\sim}(\Delta, u : A) \vdash c^\circ : p$	Assumption
2	$\Gamma^\circ, {\sim}\Delta, u : {\sim}A^\circ \vdash c^\circ : p$	Def. of $\sim$ on contexts
3	$\Gamma^\circ, {\sim}\Delta \vdash \lambda u : {\sim}A^\circ.\, c^\circ : {\sim}A^\circ \to p$	$\to$I
4	$\Gamma^\circ, {\sim}\Delta \vdash \lambda u : {\sim}A^\circ.\, c^\circ : {\sim}{\sim}A^\circ$	Def. of $\sim$ on types
5	$\Gamma^\circ, {\sim}\Delta \vdash \mathrm{dne}_A(\lambda u : u : {\sim}A^\circ.\, c^\circ) : A^\circ$	$\to$E

So we define

$$(\mu u : A.\, c)^\circ = \mathrm{dne}_A(\lambda u : {\sim}A^\circ.\, c^\circ)$$

Translating Refutation by Contradiction

$$\frac{\Gamma, x : A; \Delta \vdash c \text{ contr}}{\Gamma; \Delta \vdash \mu x : A.\, c : A \text{ false}}$$

1. We assume $\Gamma, x : A^\circ, \sim\Delta \vdash c^\circ : p$
2. So $\Gamma^\circ, x : A^\circ, \sim\Delta \vdash c^\circ : p$
3. So $\Gamma^\circ, \sim\Delta \vdash \lambda x : A^\circ.\, c^\circ : A^\circ \to p$
4. So $\Gamma^\circ, \sim\Delta \vdash \lambda x : A^\circ.\, c^\circ : \sim A^\circ$

So we define

$$(\mu x : A.\, c)^\circ = \lambda x : A^\circ.\, c^\circ$$

Consequences

- We now have a proof that every classical proof has a corresponding intuitionistic proof
- So classical logic is a *subsystem* of intuitionistic logic
- Because intuitionistic logic is consistent, so is classical logic
- Classical logic can inherit operational semantics from intuitionistic logic!

Many Different Embeddings

- Many different translations of classical logic were discovered many times
    - Gerhard Gentzen and Kurt Gödel
    - Andrey Kolmogorov
    - Valery Glivenko
    - Sigekatu Kuroda
- The key property is to show that $\sim\sim A^{\circ} \to A^{\circ}$ holds.

The Gödel-Gentzen Translation

Now, we can define a translation on types as follows:

$$\begin{array}{lcl} \neg A^\circ & = & {\sim}A^\circ \\ \top^\circ & = & 1 \\ (A \wedge B)^\circ & = & A^\circ \times B^\circ \\ \bot^\circ & = & p \\ (A \vee B)^\circ & = & {\sim}({\sim}A^\circ \times {\sim}B^\circ) \end{array}$$

- This uses a different de Morgan duality for disjunction

The Kolmogorov Translation

Now, we can define another translation on types as follows:

$$\begin{array}{lcl}
\neg A^\bullet & = & {\sim}{\sim}{\sim} A^\bullet \\
A \supset B^\bullet & = & {\sim}{\sim}(A^\bullet \to B^\bullet) \\
\top^\bullet & = & {\sim}{\sim}1 \\
(A \wedge B)^\bullet & = & {\sim}{\sim}(A^\bullet \times B^\bullet) \\
\bot^\bullet & = & {\sim}{\sim}\bot \\
(A \vee B)^\bullet & = & {\sim}{\sim}(A^\bullet + B^\bullet)
\end{array}$$

- Uniformly stick a double-negation in front of each connective.
- Deriving ${\sim}{\sim}A^\bullet \to A^\bullet$ is particularly easy:
    - The tne term will always work!

Implementing Classical Logic Axiomatically

- The proof theory of classical logic is elegant
- It is also very awkward to use:
    - Binding only arises from proof by contradiction
    - Difficult to write nested computations
    - Continuations/stacks are always explicit
- Functional languages make the stack implicit
- Can we make the continuations implicit?

The Typed Lambda Calculus with Continuations

$$
\begin{array}{llcl}
\text{Types} & X & ::= & 1 \mid X \times Y \mid 0 \mid X + Y \mid X \to Y \mid \neg X \\
\text{Terms} & e & ::= & x \mid \langle\rangle \mid \langle e, e \rangle \mid \mathsf{fst}\, e \mid \mathsf{snd}\, e \\
 & & \mid & \mathsf{abort} \mid \mathsf{L}\, e \mid \mathsf{R}\, e \mid \mathsf{case}(e, \mathsf{L}\, x \to e', \mathsf{R}\, y \to e'') \\
 & & \mid & \lambda x : X.\, e \mid e\, e' \\
 & & \mid & \mathsf{throw}(e, e') \mid \mathsf{letcont}\, x.\, e \\
\text{Contexts} & \Gamma & ::= & \cdot \mid \Gamma, x : X
\end{array}
$$

Units and Pairs

$$\frac{}{\Gamma \vdash \langle\rangle : 1}\ 1\mathsf{I}$$

$$\frac{\Gamma \vdash e : X \qquad \Gamma \vdash e' : Y}{\Gamma \vdash \left\langle e, e' \right\rangle : X \times Y}\ \times\mathsf{I}$$

$$\frac{\Gamma \vdash e : X \times Y}{\Gamma \vdash \mathsf{fst}\, e : X}\ \times\mathsf{E}_1 \qquad\qquad \frac{\Gamma \vdash e : X \times Y}{\Gamma \vdash \mathsf{snd}\, e : Y}\ \times\mathsf{E}_1$$

Functions and Variables

$$\frac{X : X \in \Gamma}{\Gamma \vdash x : X}\ \textsc{Hyp} \qquad\qquad \frac{\Gamma, x : X \vdash e : Y}{\Gamma \vdash \lambda x : X.\, e : X \to Y}\ {\to}\mathrm{I}$$

$$\frac{\Gamma \vdash e : X \to Y \qquad \Gamma \vdash e' : X}{\Gamma \vdash e\, e' : Y}\ {\to}\mathrm{E}$$

Sums and the Empty Type

$$\frac{\Gamma \vdash e : X}{\Gamma \vdash \mathsf{L}\, e : X + Y} +\mathsf{I}_1 \qquad \frac{\Gamma \vdash e : Y}{\Gamma \vdash \mathsf{R}\, e : X + Y} +\mathsf{I}_2$$

$$\frac{\Gamma \vdash e : X + Y \qquad \Gamma, x : X \vdash e' : Z \qquad \Gamma, y : Y \vdash e'' : Z}{\Gamma \vdash \mathsf{case}(e, \mathsf{L}\, x \to e', \mathsf{R}\, y \to e'') : Z} +\mathsf{E}$$

(no intro for 0) $$\frac{\Gamma \vdash e : 0}{\Gamma \vdash \mathsf{abort}\, e : Z} 0\mathsf{E}$$

Continuation Typing

$$\frac{\Gamma, u : \neg X \vdash e : X}{\Gamma \vdash \mathrm{letcont}\, u : X.\ e : X}\ \textsc{Cont} \qquad \frac{\Gamma \vdash e : \neg X \qquad \Gamma \vdash e' : X}{\Gamma \vdash \mathrm{throw}_Y(e, e') : Y}\ \textsc{Throw}$$

Examples

Double-negation elimination:

$$\begin{array}{lcl} \mathrm{dne}_X & : & \neg\neg X \to X \\ \mathrm{dne}_X & \triangleq & \lambda k : \neg\neg X.\ \mathrm{letcont}\, u : \neg X.\ \mathrm{throw}(k, u) \end{array}$$

The Excluded Middle:

$$\begin{array}{l} t : X \vee \neg X \\ t \triangleq \mathrm{letcont}\, u : \neg(X \vee \neg X). \\ \qquad \mathrm{throw}(u, \mathrm{R}\,(\mathrm{letcont}\, q : \neg\neg X. \\ \qquad\qquad\qquad \mathrm{throw}(u, \mathrm{L}\,(\mathrm{dne}_X\, q))) \end{array}$$

Continuation-Passing Style (CPS) Translation

Type translation:

$$
\begin{array}{lcl}
\neg X^{\bullet} & = & \sim\sim\sim X^{\bullet} \\
X \to Y^{\bullet} & = & \sim\sim(X^{\bullet} \to Y^{\bullet}) \\
1^{\bullet} & = & \sim\sim 1 \\
(X \times Y)^{\bullet} & = & \sim\sim(X^{\bullet} \times Y^{\bullet}) \\
0^{\bullet} & = & \sim\sim 0 \\
(X + Y)^{\bullet} & = & \sim\sim(X^{\bullet} + Y^{\bullet})
\end{array}
$$

Translating contexts:

$$
\begin{array}{lcl}
(\cdot)^{\bullet} & = & \cdot \\
(\Gamma, x : A)^{\bullet} & = & \Gamma^{\bullet}, x : A^{\bullet}
\end{array}
$$

Theorem If $\Gamma \vdash e : X$ then $\Gamma^\bullet \vdash e^\bullet : X^\bullet$.

Proof: By induction on derivations – we "just" need to define $e^\bullet$.

The CPS Translation

$$
\begin{array}{lcl}
x^{\bullet} & = & \lambda k.\, x\, k \\
\langle\rangle^{\bullet} & = & \lambda k.\, k\ \langle\rangle \\
\langle e_1, e_2\rangle^{\bullet} & = & \lambda k.\, e_1^{\bullet}\, (\lambda x.\, e_2^{\bullet}\, (\lambda y.\, k\, (x, y))) \\
(\mathsf{fst}\, e)^{\bullet} & = & \lambda k.\, e^{\bullet}\, (\lambda p.\, k\, (\mathsf{fst}\, p)) \\
(\mathsf{snd}\, e)^{\bullet} & = & \lambda k.\, e^{\bullet}\, (\lambda p.\, k\, (\mathsf{snd}\, p)) \\
(\mathsf{L}\, e)^{\bullet} & = & \lambda k.\, e^{\bullet}\, (\lambda x.\, k\, (\mathsf{L}\, x)) \\
(\mathsf{R}\, e)^{\bullet} & = & \lambda k.\, e^{\bullet}\, (\lambda y.\, k\, (\mathsf{R}\, y)) \\
\mathsf{case}(e, \mathsf{L}\, x \to e_1, \mathsf{R}\, y \to e_2)^{\bullet} & = & \lambda k.\, e^{\bullet}\, (\lambda v.\, \mathsf{case}(v, \\
 & & \qquad\qquad\quad \mathsf{L}\, x \to e_1^{\bullet}\, k \\
 & & \qquad\qquad\quad \mathsf{R}\, y \to e_2^{\bullet}\, k) \\
(\lambda x : X.\, e)^{\bullet} & = & \lambda k.\, k\, (\lambda x : X^{\bullet}.\, e^{\bullet}) \\
(e_1\, e_2)^{\bullet} & = & \lambda k.\, e_1^{\bullet}\, (\lambda f.\, e_2^{\bullet}\, (\lambda x.\ k\, (f\, x)))
\end{array}
$$

The CPS Translation for Continuations

$$
\begin{aligned}
(\mathrm{letcont}\, u : \neg X.\, e)^{\bullet} &= \lambda k.\, [(\lambda q.\, q\, k)/u](e^{\bullet}) \\
\mathrm{throw}(e_1, e_2)^{\bullet} &= \mathrm{tne}(e_1^{\bullet})\, e_2^{\bullet}
\end{aligned}
$$

- The rest of the CPS translation is bookkeeping to enable these two clauses to work!

Questions

1. Give the embedding (ie, the e° and k° translations) of classical into intuitionistic logic for the Gödel-Gentzen translation. You just need to give the embeddings for sums, since that is the only case different from lecture.
2. Using the intuitionistic calculus extended with continuations, give a typed term proving *Peirce's law*:

$$((X \to Y) \to X) \to X$$