Topics in Concurrency

Lectures 4

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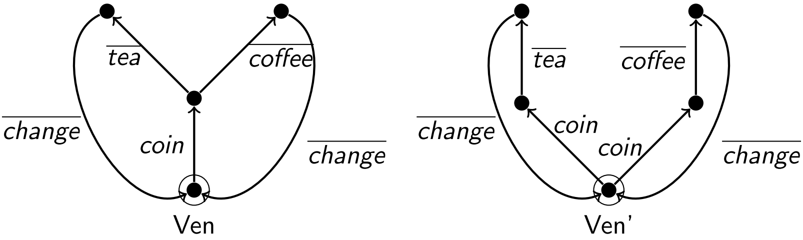
Two vending machine implementations



$$User \stackrel{\text{def}}{=} \overline{coin}.coffee.change.\overline{work}$$

Specification and correctness:

- Assertions and logic (e.g. $(User \parallel Ven) \setminus \{coin, change, coffee, tea\}$ always outputs *work*)
- Equivalence

Language equivalences

- A trace of a process p is a (possibly infinite) sequence of actions

$$(a_1, a_2, \ldots, a_i, a_{i+1}, \ldots)$$

  such that

$$p \xrightarrow{a_1} p_1 \xrightarrow{a_2} \ldots p_{i-1} \xrightarrow{a_i} p_i \xrightarrow{a_{i+1}} \ldots$$

- Two processes are trace equivalent iff they have the same sets of traces

- Are Ven and Ven' trace equivalent?
- Are $(User \parallel Ven) \setminus \{coin, change, coffee, tea\}$ and $(User \parallel Ven') \setminus \{coin, change, coffee, tea\}$ trace equivalent?

Completed trace equivalence

- A trace is maximal if it cannot be extended (it is either infinite or ends in a state from which there is no transition)
- Processes are completed trace equivalent iff they have the same sets of *maximal* traces.

- Are Ven and Ven' completed trace equivalent?
- Are $(User \parallel Ven) \setminus \{coin, change, coffee, tea\}$ and $(User \parallel Ven') \setminus \{coin, change, coffee, tea\}$ completed trace equivalent?

A more subtle form of equivalence is needed to reason compositionally about processes

Bisimulation — a process equivalence

To

- support equational reasoning
- simplify verification

Strong bisimulation

A (strong) bisimulation is a relation R between states for which
If $p \; R \; q$ then:

1. $\forall \alpha, p'. \quad p \xrightarrow{\alpha} p' \quad \Longrightarrow$
   $\exists q'. \quad q \xrightarrow{\alpha} q' \quad \& \quad p' \; R \; q'$
2. $\forall \alpha, q'. \quad q \xrightarrow{\alpha} q' \quad \Longrightarrow$
   $\exists p'. \quad p \xrightarrow{\alpha} p' \quad \& \quad p' \; R \; q'$

(Strong) bisimilarity is an equivalence on states

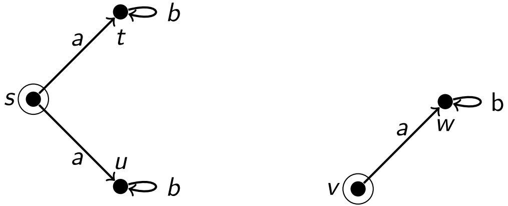
$$p \sim q \qquad \textit{iff} \qquad p \; R \; q \text{ for some (strong) bisimulation } R$$

Exhibiting bisimilarity

To show $p_1 \sim p_2$, we give a relation R such that R is a bisimulation and $p_1 \; R \; p_2$.

Examples: Give bisimulations to show

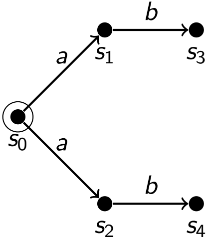
- $a \parallel b \sim a.b + b.a$
- On transition systems, $s \sim v$ where



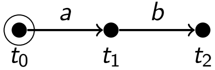
Examples: Looping

$$\begin{tikzcd} s_0 \arrow[loop above, "a"] & ?\sim? & t_0 \arrow[r, "a"] & t_1 \arrow[r, "a"] & t_2 \arrow[r, "a"] & t_3 \arrow[r, "a"] & \cdots \end{tikzcd}$$

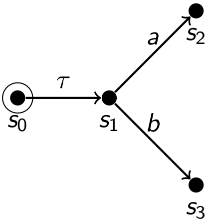
Examples: Inessential branching



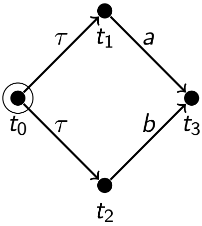
$? \sim ?$



Examples: Internal choice



$? \sim ?$



Bisimulations

If R, S, R_i for $i \in I$ are strong bisimulations then so are:

1. Id, the identity relation the set of states of any transition system
2. R^{op}, the converse/opposite relation
3. $R \circ S$, the composition (when the transition systems involved match up so that the composition makes sense)
4. $\bigcup_{i \in I} R_i$, the union (when the relations are over the same transition systems)

(1)–(3) imply that $\sim$ is an equivalence relation, and (4) that $\sim$ is a bisimulation.

Equational properties of bisimulation

$+$ and $\parallel$ are commutative and associative w.r.t. $\sim$, with unit **nil**

If $p \sim q$ then:

- $\alpha.p \sim \alpha.q$
- $p + r \sim q + r$
- $p \parallel r \sim q \parallel r$
- $p \setminus L \sim q \setminus L$
- $p[f] \sim q[f]$

... bisimilarity is a congruence

Expansion laws for CCS

In general,

$$p \sim \sum \{\alpha.p' \mid p \xrightarrow{\alpha} p'\}$$

We can use this to remove everything but prefixing and sums:

Suppose $p \sim \sum_{i \in I} \alpha_i.p_i$ and $q \sim \sum_{j \in J} \beta_j.q_j$.

$$\begin{array}{rcl} p \setminus L & \sim & \sum \{\alpha_i.(p_i \setminus L) \mid \alpha_i \notin L\} \\ p[f] & \sim & \sum \{f(\alpha_i).(p_i[f]) \mid i \in I\} \\ p \parallel q & \sim & \sum_{i \in I} \alpha_i.(p_i \parallel q) + \sum_{j \in J} \beta_j.(p \parallel q_j) \\ & + & \sum \{\tau.(p_i \parallel q_j) \mid \alpha_i = \overline{\beta_j}\} \end{array}$$

Strong bisimilarity and specifications

An example:

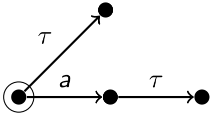
$$\begin{aligned}
Sem &\stackrel{\text{def}}{=} get.put.Sem \\[1em]
P_1 &\stackrel{\text{def}}{=} \overline{get}.a_1.b_1.\overline{put}.P_1 \\
P_2 &\stackrel{\text{def}}{=} \overline{get}.a_2.b_2.\overline{put}.P_2 \\[1em]
Sys &\stackrel{\text{def}}{=} (Sem \parallel P_1 \parallel P_2) \setminus \{get, put\} \\[1em]
Spec &\stackrel{\text{def}}{=} \tau.a_1.b_1.Spec + \tau.a_2.b_2.Spec \\
Spec' &\stackrel{\text{def}}{=} \tau.a_1.b_1.\tau.Spec' + \tau.a_2.b_2.\tau.Spec'
\end{aligned}$$

Do we have

$$? \quad Sys \sim Spec \quad ? \quad \text{or} \quad ? \quad Sys \sim Spec' \quad ?$$

Weak bisimulation

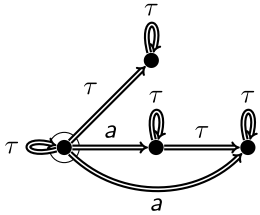
Hiding τ actions



$$\overset{\tau}{\Rightarrow} \overset{\mathrm{def}}{=} (\overset{\tau}{\rightarrow}^{*})$$

$$\overset{a}{\Rightarrow} \overset{\mathrm{def}}{=} (\overset{\tau}{\Rightarrow}\overset{a}{\rightarrow}\overset{\tau}{\Rightarrow})$$

We get a transition system



Weak bisimulation is bisimulation w.r.t. $\Rightarrow$

Weak bisimulation

A weak bisimulation is a relation R between states for which
If $p\ R\ q$ then:

1. $\forall \alpha, p'. \quad p \stackrel{\alpha}{\Rightarrow} p' \quad \Longrightarrow$
   $\exists q'. \quad q \stackrel{\alpha}{\Rightarrow} q' \quad \& \quad p'\ R\ q'$
2. $\forall \alpha, q'. \quad q \stackrel{\alpha}{\Rightarrow} q' \quad \Longrightarrow$
   $\exists p'. \quad p \stackrel{\alpha}{\Rightarrow} p' \quad \& \quad p'\ R\ q'$

Weak bisimulation is not a congruence $\rightsquigarrow$ observational congruence.