Probability and Computation: Problem sheet 5

You are encouraged to submit your solutions by emailing them to luca.zanetti@cl.cam.ac.uk by Wednesday 27th of February.

Question 1. *Recall that $\tau(\epsilon)$ refers to the mixing time in total variation distance, while $\tau_2(\epsilon)$ refers to the mixing time in ℓ_2-norm. Prove that,*

$$\tau(2\epsilon) \leq \tau_2(\epsilon).$$

To prove it you might need Jensen's inequality: $\psi(\mathbf{E}[\, X\,]) \leq \mathbf{E}[\, \psi(X)\,]$ for any convex function ψ.

We have also mentioned in class that $\tau_2(\epsilon) = O(\tau(2\epsilon)\log(1/\pi_*))$, where $\pi_* = \min_x \pi(x)$. Can you find an example where this logarithmic factor is needed?

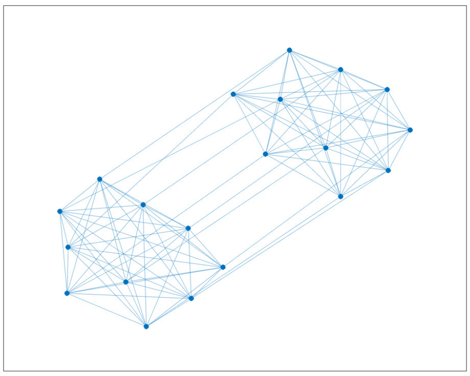
Question 2. *The d-dimensional hypercube is an undirected graph $G = (V, E)$ such that V can be represented as the set of binary strings of length d, i.e., $V = \{0, 1\}^d$, and $\{x, y\} \in E$ if and only if x and y differ by exactly one bit.*

1. *Compute $|V|$ and $|E|$.*
2. *Prove that G has conductance $O(1/\log(|V|))$.*

Question 3. *Let G be a graph of $2n$ vertices that consists in two complete graphs of n vertices, connected by a perfect matching (see picture below). Notice this is a regular graph of degree n and its edge-set can be described as follows:*

$$E = \{\{u, v\} : u, v \in \{1, \ldots, n\} \text{ s.t. } u \neq v\} \bigcup \{\{u, v\} : u, v \in \{n+1, \ldots, 2n\} \text{ s.t. } u \neq v\}$$
$$\bigcup \{\{u, v\} : v = u + n, \text{ for } u = 1, \ldots, n\}.$$

1. *What is the conductance of G?*
2. *Prove that $1 - \lambda_2 = O(1/n)$, which implies that the mixing time of G is at least $\Omega(n)$.*



Question 4. *Use the variational characterisation to show that a cycle on n vertices has spectral gap $1 - \lambda = O(n^{-2})$. To obtain such a result, we suggest you look at the proof for the lower bound of the spectral gap in regular graphs and understand why such a proof gives us an asymptotically tight result for the cycle.*

Question 5. *Prove the claim made in Lecture 10: for a regular undirected graph with degree d and diameter δ,*

$$d \cdot \delta = O(n).$$

Question 6. *(Bonus) In this exercise you are asked to prove the variational characterisation of the spectral gap. Let P be a self-adjoint transition matrix with stationary distribution π and eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$.*

1. *The first step consists mainly in a few algebraic manipulations. Prove that, for any nonzero $f : \Omega \to \mathbb{R}$,*

$$\langle f, (I - P)f \rangle_\pi = \frac{1}{2} \sum_{x,y \in \Omega} (f(x) - f(y))^2 P(x, y)\pi(x)$$

2. *Now use the spectral theorem to prove that, for any f such that $f \perp 1$,*

$$\langle f, (I - P)f \rangle_\pi \geq (1 - \lambda_2)\|f\|_{2,\pi}^2$$

3. *Finally, prove that for an eigenvector f_2 with eigenvalue λ_2 and unit norm,*

$$\langle f_2, (I - P)f_2 \rangle_\pi = 1 - \lambda_2$$

   and argue that this proves the variational characterisation.

4. *Modify the proof above to show a variational characterisation for λ_n:*

$$1 - \lambda_n = \max_{f \neq 0} \frac{\sum_{x,y \in \Omega}(f(x) - f(y))^2 P(x, y)\pi(x)}{2\|f\|_{2,\pi}^2}$$

Question 7. *Use the variational characterisations proved in the previous exercise to solve Question 5 of Problem Sheet 4.*

Hints

Q2.2: Look at the "dimension cut" $S = \{x \in V : x_1 = 0\}$.

Q3.2: Construct a function f which is constant on each complete subgraph, but has opposite sign on the two. Use this f to obtain an upper bond on the spectral gap using the variational characterisation.

Q6.2: First deduce from the spectral theorem (more precisely from the corollaries seen in the lecture), that for any $f \perp 1$, $\|f\|_{2,\pi}^2 = \sum_{i=2}^n \langle f, f_i \rangle_\pi^2 f_i$, where $\{1, f_2, \ldots, f_n\}$ is an orthonormal set of eigenvectors for P. You also need to use the linearity of the inner product.