Probability and Computation: Problem sheet 4

You are encouraged to submit your solutions by emailing them to luca.zanetti@cl.cam.ac.uk by Wednesday 20th of February.

Question 1. *Let $(X_i)_{i\geq 1}$ be independent random variables with $\mathbf{P}[\, X_i = 1\,] = \mathbf{P}[\, X_i = -1\,] = 1/2$. Let $S_n = \sum_{i=0}^{n} X_n$, with $X_0 = K > 0$. For $N > K$ define*

$$T = T_{0,N} = \min\{n \geq 0 : S_n = 0 \text{ or } S_n = N\}.$$

1. *Prove that $\mathbf{E}[\, T\,] < \infty$ (you cannot use the OST).*
2. *Find a deterministic sequence of values $a_n \in \mathbb{R}$ such that $Z_n = S_n^3 + a_n S_n$ is a martingale w.r.t. $X_0, X_1, \ldots$*
3. *Find deterministic sequences $b_n, c_n \in \mathbb{R}$ such that $W_n = S_n^4 + b_n S_n^2 + c_n$ is a martingale w.r.t. $X_0, X_1, \ldots$*

Question 2.

1. *Consider out path on vertices $\{0, \ldots, N\}$, and suppose $X_0 = K$. Compute $h_{K,N}$*
2. *Compute the cover time of a path on $\{0, \ldots, N\}$ when N is even. What about when N is odd?*
3. *Consider a cycle on vertices $\{0, 1, \ldots, N\}$ where vertex i is adjacent to $i + 1$, and 0 is adjacent to N. Compute the cover time.*
4. *Consider a cycle on vertices $\{0, 1, \ldots, N\}$. Define T by*

$$T = \min\{m : \cup_{i=0}^{m} X_i = \{0, \ldots, N\}\}$$

T is the first time all vertices has been covered. Compute $\mathbf{P}[\, X_T = i | X_0 = 0\,]$ for $i \in \{1, \ldots, N\}$.

Question 3. ***Wald's Equation:*** *Let $X_1, \ldots,$ i.i.d. non-negative random variables with finite expectation. Let T be a stopping time with respect to this sequence and suppose that $\mathbf{E}[\, T\,] < \infty$ and that $\mathbf{E}[\, |X_1|\,] < \infty$. Prove that*

$$\mathbf{E}\left[\sum_{i=1}^{T} X_i\right] = \mathbf{E}[\, T\,]\, \mathbf{E}[\, X_1\,].$$

Question 4. *A weighted undirected graph $G = (V, E, w)$ is defined by a set vertices V, a collection of edges $E \subseteq V \times V$, and a weight function $w \colon V \times V \to \mathbb{R}_{\geq 0}$ such that, for any $u, v \in V$, $w(u, v) = w(v, u)$ and $w(u, v) > 0$ if and only if $(u, v) \in E$. Self-loops of the kind (u, u) are allowed. A random walk on $G = (V, E, w)$ is a Markov chain with transition matrix P such that, for any $u, v \in V$, $P(u, v) = w(u, v)/d(u)$, where $d(u) = \sum_{z \in V} w(u, z)$.*

1. *What is the stationary distribution of this Markov chain?*
2. *What does being aperiodic amounts to?*
3. *Prove that a Markov chain is reversible if and only if it can be represented by a random walk on a weighted undirected graph.*
4. *Prove that if P is reversible, then P^t is also reversible for any $t \in \mathbb{N}$.*

Question 5. *Let P be the transition matrix of a (simple) random walk on an undirected graph $G = (V, E)$. Let $\lambda_1 \geq \cdots \geq \lambda_n$. Prove the following.*

1. $\lambda_1 = 1$.

2. $\lambda_2 = 1$ *if and only if the graph is disconnected.*

3. $\lambda_n = -1$ *if and only if there exists a bipartite connected component.*

4. *Suppose now that the random walk is lazy (i.e.,* $P(u, u) \geq 1/2$ *for any* $u \in V$*). Prove that all the eigenvalues of* P *are non-negative.*