Probability and Computation: Problem sheet 3

Question 1. *We are going to prove part of nicer version of the Chernoff Bounds. Prove the following inequalities*

i) For $0 < \delta < 1$,

$$\frac{e^{\delta}}{(1+\delta)^{(1+\delta)}} \leq e^{-\delta^2/3}.$$

ii) For $0 < \delta < 1$,

$$\frac{e^{-\delta}}{(1-\delta)^{(1-\delta)}} \leq e^{-\delta^2/2}.$$

iii) Using the Chernoff-Bounds (Slides 8 and 22, Lecture 5) deduce the second part of the Nicer Chernoff Bounds in Slide 23 of Lecture 5.

Question 2. *Chernoff Bounds for other random variables.*

i) Let X *be a Poisson random variable of mean* μ. *Prove that*

$$\mathbf{E}[\, e^{\lambda X} \,] = e^{\mu(e^{\lambda}-1)}$$

and deduce that for $t \geq \mu$ *and for* $\delta \in (0,1)$

$$\mathbf{P}[\, X \geq t \,] \leq e^{-\mu}\left(\frac{e\mu}{t}\right)^t \quad \textit{and} \quad \mathbf{P}[\, X \geq (1+\delta)\mu \,] \leq e^{-\delta^2\mu/3},$$

and the corresponding lower tails.

ii) Let X *be a Normal random variable of mean* μ *and variance* σ^2. *Prove that*

$$\mathbf{E}[\, e^{\lambda X} \,] = e^{\mu\lambda + \sigma^2\lambda^2/2},$$

and deduce that for $t > \mu$

$$\mathbf{P}[\, X \geq t \,] \leq \exp\left[\frac{-(t-\mu)^2}{2\sigma^2}\right].$$

Find the corresponding lower tail.

Question 3. *Show properties 2-6 of slide 19 of Lecture 6.*

Question 4. *Let* $X_1, \ldots, X_n$ *be independent discrete random variables and let* $Z = f(X_1, \ldots, X_n)$ *for some function* f. *Prove that*

$$\mathbf{E}[\, Z | X_1, \ldots, X_i \,] = \sum_{x_i, x_{i+1}, \ldots, x_n} f(X_1, \ldots, X_i, x_i, \ldots, x_n)\mathbf{P}[\, X_i = x_i, \ldots, X_n = x_n \,]$$

Question 5. *Conditional Variance. Define the conditional variance of* Y *given* X *as*

$$\mathbf{Var}\,[\, Y|X \,] = \mathbf{E}[\, (Y - \mathbf{E}[\, Y|X \,])^2 | X \,].$$

1. *Prove that* $\mathbf{Var}\,[\, Y \,] = \mathbf{E}[\, \mathbf{Var}\,[\, Y|X \,] \,] + \mathbf{Var}\,[\, \mathbf{E}[\, Y|X \,] \,]$
2. *Consider* n *bins and a random number* M *of balls, where* $\mathbf{E}[\, M \,] = \mu$ *and* $\mathbf{Var}\,[\, M \,] = \sigma^2$. *Compute the variance of the number of balls that are assigned to the first bin.*

Question 6. *Consider a coin that shows head with probability* p. *What is the expected number of flips required to observe a run of* n *consecutive heads?*

Question 7. *Let $X_1, \ldots, X_n$ i.i.d. samples from a distribution of interest. We know that $\mathbf{E}[X_i] = \mu$ and $\mathbf{Var}[X_i] = \sigma^2$ for all i, but we do not know the exact values of μ nor σ^2. We are given the mission to find an estimate $\hat{\mu}$ of the actual mean μ. We want the estimate $\hat{\mu}$ to satisfy the (δ, ε) condition: given ε, we want that $\hat{\mu} \in [\mu - \varepsilon\sigma, \mu + \varepsilon\sigma]$ with probability at least $1 - \delta$. How many data points X_i do we need to build an estimator satisfying the (δ, ε) condition?*

- *In a first attempt we can just deliver $\hat{\mu} = \frac{\sum_{i=1}^{n} X_i}{n}$, nevertheless, we cannot guarantee a good behaviour of such estimator, as we do not have enough information to compute a Chernoff Bound for it.*

1. *Prove that with $m = \lceil \frac{10}{\varepsilon^2} \rceil$ data points, we have that $\hat{\mu}_m = \left(\sum_{i=1}^{m} X_i\right)/m$ satisfies the $(1/10, \varepsilon)$ condition.*
2. *Write an algorithm that uses at most $O\left(\frac{\log(\delta^{-1})}{\varepsilon^2}\right)$ data points to build an estimate of μ satisfying the (δ, ε) condition.*

Hint.

Q6: *Recall how we deduce the expectation of a geometric in class.*

Q7: *For 2. consider batches of size $m = \lceil \frac{10}{\varepsilon^2} \rceil$. What can you say about more than half of them?*