Probability and Computation: Problem sheet 2

Question 1. *A Lion and a Deer each independently take a random walk on a connected, undirected, non-bipartite graph G. They start at the same time on different nodes, and each makes one transition at each time step. The Lion eats the Deer if they are ever at the same node at some time step. Let n and m denote, respectively, the number of vertices and edges of G. Show an upper bound of $O\big(m^2n\big)$ on the expected time before the cat eats the mouse.*

Question 2. *Let $t_{hit}(G) = \max_{x,y\in V} \mathbf{E}_x\left[\tau_y^+\right]$ be the maximum hitting time. Prove the following weak version of the Matthew's bound: For any graph G we have $t_{cov}\,(\,G\,) = O(t_{hit} \cdot \log n)\,.$*

Question 3. *Let X and Y be two Binomial random variables with parameters (n, p_1) and (n, p_2) respectively, for a positive integer n and for $0 \le p_1 \le p_2 \le 1$. Show that for any k*

$$\mathbf{P}[\,X \ge k\,] \le \mathbf{P}[\,Y \ge k\,]\,.$$

(i) Define a function $f(p) = \mathbf{P}[\,B(n, p) \ge k\,]$ and show that it is increasing in p.

(ii) Now use coupling to prove the same statement, extending the coupling of two unfair coins given in the lecture.

Question 4. *Consider an infinite ladder with steps numbered starting from 0. When at step n, we move to step $(n + 1)$ with probability p_n or fall to step 0 with probability $1 - p_n$. The sequence p_n is decreasing (we are being more and more cautious). Prove that $\mathbf{P}[\,X_n > k|X_0 = j\,]$ is an increasing function of j.*

Question 5. *We saw the Top-to-Random shuffle in lectures. One can also shuffle cards by doing the reverse of this process, the Random-to-Top Shuffle: at each time step a card is taken at random from the deck and placed on the top.*

(i) Construct a coupling between two copies of the R-to-T chain so that after some (random) time they will be in the same state.

(ii) Apply the Coupling Lemma for Mixing to your coupling to bound the mixing time of the R-to-T Chain.

Question 6. *Consider the Balls-into-Bins setting where we have n balls and n bins. We assume that n is large enough. We are going to prove that the maximum load is whp. at least $c \log n/ \log\log n$ for some $c > 0$.*

(i) Let $Y_j(k)$ be the random variable that indicates that bin j receives at least k balls. Prove that for any $k \le n$, it holds that

$$\mathbf{P}[\,Y_j(k) = 1\,] \ge \frac{e^{-2}}{k^k}.$$

(ii) Show it exist $c > 0$ such that for $k^* = \left\lfloor \frac{c \log n}{\log\log n} \right\rfloor$, we have

$$\mathbf{P}[\,Y_j(k^*) = 1\,] \ge n^{-1/3},$$

(iii) Argue that for any $k \le n$, and any bins i, j we have

$$\mathbf{P}[\,Y_j(k)Y_i(k) = 1\,] \le \mathbf{P}[\,Y_i(k) = 1\,]\,\mathbf{P}[\,Y_j(k) = 1\,]$$

(iv) Let $Y = \sum_{j=1}^{n} Y_j(k^*)$. Check that $\mathbf{E}[\,Y\,] \ge n^{2/3}$ and that $\mathbf{Var}\,[\,Y\,] \le n$

(v) Conclude that $\mathbf{P}[\,Y = 0\,] \le n^{-1/3}$

Hint (Collected hints for the exercises).

Q1: *Consider a Markov chain whose states are the ordered pairs (a, b), where a is the position of the Lion and b is the position of the Deer.*

Q2: *Use Markov's Inequality.*

Q3: *Hint 1:* $\sum_{i=k}^{n} a_i \geq \sum_{i=0}^{n} a_i$ *if the negative terms of the sum are all in the beginning.*
Hint 2: $\sum_{i=0}^{n} \binom{n}{i} i p^{i-1}(1-p)^{n-i-1} = 0$. *Why?*

Q5(i): *Run one copy of the chain X_t as normal, for the other make sure that at each time step the same card as in X_t is on the top.*

Q5(iii): *Similar to Balls and bins makes an appearance here.*

Q6(i): *Hint 1: Compute of getting exactly k balls in the bin.*
Hint 2: Use that $1 - x \geq e^{-2x}$ for $x \in [0, 1/2]$ and that $\binom{n}{k} \geq (n/k)^k$.