Probability and Computation: Problem sheet 1

Question 1. *Recall that a permutation* $\sigma : [n] \to [n]$ *is a bijection from* $[n]$ *to* $[n]$*. A cycle* $c = (c_1, c_2, \ldots, c_k)$ *is a sequence such that* $\sigma(a_i) = a_{i+1 \mod k}$ *. Example: if* $\sigma(1) = 3$*,* $\sigma(2) = 2$*,* $\sigma(3) = 4$ *and* $\sigma(4) = 1$ *(this is also denoted* $\sigma = \binom{1\,2\,3\,4}{3\,2\,4\,1}$ *) then* $(1,3,4)$ *is a cycle (as are* $(3,4,1)$ *and* $(4,1,3)$*) .*

(i) How many permutations on $[n]$ *are there?*

(ii) Show that the number of permutations on $[n]$ *which contain a cycle of length* ℓ *is* $n!/\ell$ *.*

(iii) Let $\mathcal{E}$ *be the event that a uniformly distributed random permutation contains no cycle longer than* $\lfloor n/2 \rfloor$*. Show that* $\mathbf{P}[\,\mathcal{E}\,] = 1 - \left(\frac{1}{\lfloor n/2 \rfloor + 1} + \ldots + \frac{1}{n}\right)$.

(iv) Approximate the expression for $\mathbf{P}[\,\mathcal{E}\,]$ *above by integrals to give* $\mathbf{P}[\,\mathcal{E}\,] = 1 - \ln 2 + o(1) \approx 0.31183$.

Question 2. *Recall that a probability vector (distribution) is a non-negative real vector whose elements sum to* 1*. A stochastic matrix is a real square matrix, where each row is a probability vector. Observe every Stochastic matrix gives rise to a Markov chain and visa versa.*

(i) Let $\nu \in \mathbb{R}_+^n$ *be a probability vector and* $M \in \mathbb{R}_+^{n \times n}$ *be a stochastic matrix. Show that* νM *is a probability vector.*

A doubly stochastic matrix is a real square matrix, where each row and column is a probability vector.

(ii) Prove that the uniform distribution is stationary for any Markov chain whose transition matrix is doubly stochastic.

Question 3. *In this question we shall consider two probability measures* μ_1, μ_2 *on the common state space* $[12] = \{1, \ldots, 12\}$*. The first measure is the uniform measure* $\mu_1(x) = 1/12$ *for all* $x \in [12]$*, this is a "12 sided fair die". The second* μ_2 *is the measure generated by the sum of two independent, fair, 6-sided die i.e.* $\mu_2(1) = 0, \mu_2(2) = 1/6^2, \ldots$*. Calculate* $\|\mu_1 - \mu_2\|$*.*

Question 4. *Recall Jerry's Coupling* Z_t *for two Simple Random walks* T_t, J_t *on the* 3*-cycle. This coupling is given by Markov chain* $Z_t = (T_t, J_t)$ *on* $\{0,1,2\} \times \{0,1,2\}$*:*

- *Run the Cat* T_t *as normal*
- *Mouse* J_t *moves according to the rule:* $J_{t+1} = \begin{cases} J_t + 1 \mod 3 & \textit{if } T_{t+1} = T_t + 1 \mod 3 \\ J_t - 1 \mod 3 & \textit{if } T_{t+1} = T_t - 1 \mod 3. \end{cases}$

This coupling lets the "Mouse" J_t *avoid the "Cat"* T_t *forever.*

1. *Write down the rule for "Tom's Coupling" - the coupling where* T_t *and* J_t *meet as soon as possible and show this is a valid coupling.*
2. *Draw a diagram of the Markov Chain* Z_t *on* $\{0,1,2\} \times \{0,1,2\}$ *generated by Tom's Coupling.*

Question 5. *In lecture* 1 *we saw the balls in bins experiment where one assigns* m *balls to* n *bins uniformly and independently. By considering the number of balls assigned to bins between the first time* i *bins are empty and* $i - 1$ *bins are empty, show that the the expected number of balls one must assign before there is no empty bin is* $n \log n + \Theta(n)$*.*

Question 6. *Let* X_n *be the sum of* n *independent rolls of a fair die. Show that, for any* $k \geq 2$*,*

$$\lim_{n \to \infty} \mathbf{P}[\, X_n \textit{ is divisible by } k\,] = \frac{1}{k}.$$

Hint. *At face value X_n is an (infinite) Markov chain on $\mathbb{N}$. We would like to consider it as a finite Markov chain, reduction $\mod m$ (for some suitable m) will help us achieve this.*

Question 7. *State j is accessible from state i if, for some integer $n \geq 0$, $P^n_{i,j} > 0$. If two states i and j are accessible from each other, we say that they communicate and we write $i \sim j$. Prove that communicating relation $\sim$ defines an equivalence relation.*

Question 8. *Prove the following Lemma from class: For any probability distributions μ and η on a countable state space Ω*

$$\|\mu - \eta\|_{tv} = \frac{1}{2} \sum_{\omega \in \Omega} |\mu(\omega) - \eta(\omega)|.$$

Hint. *Recall the sets $\Omega^{\pm}$ from the Coupling Lemma.*

Question 9. *This question asks you to prove lower bounds on the mixing time of some lazy random walks on graphs.*

1. *Let $G = (V_1 \cup V_2)$ be a graph made of two disjoint complete graphs of n vertices, supported respectively on V_1 and V_2, connected by a single edge. This is called the Barbell graph. Consider a lazy random walk on G. Prove that $t_{mix}(G) = \Omega(n^2)$ (recall from Lecture 3 that $t_{mix} = \tau(1/4)$).*

2. *Suppose now we add $s < n$ edges to the Barbell graph, where each edge has one endpoint in V_1 and the other endpoint in V_2. What happens to $t_{mix}(G)$?*

3. *Consider now a version of the Barbell graph where $|V_1| = n, |V_2| = \lfloor \log(n) \rfloor$ and there exists only an edge between V_1 and V_2. What is the mixing time of this graph?*