Lecture 8: The Optional Stopping Theorem

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Outline

Stopping Times

Optional Stopping Theorem

Applications to Random Walks

Martingales: Definition

A sequence of random variables $Z_0, Z_1, \dots,$ is a martingale with respect to the sequence $X_0, X_1, \dots,$ if, for all $n \geq 0$, the following holds:

1. Z_n is a function of $X_0, X_1, \dots, X_n$
2. $\mathbf{E}[\,|Z_n|\,] < \infty$, and
3. $\mathbf{E}[\,Z_{n+1}|X_0, \dots, X_n\,] = Z_n$.

The Gambler's Ruin Problem

- Consider a sequence $X_1, X_2, \ldots,$ of independent random variables with $\mathbf{P}[\,X_i = 1\,] = \mathbf{P}[\,X_i = -1\,] = 1/2$
- For $n \geq 1$ denote by S_n the sum $X_0 + X_1 + \ldots + X_n$, where $X_0 = K$,
- Then $S_0, S_1, \ldots,$ is a martingale with respect to $X_0, X_1, X_2, \ldots,$
- We check the definition
  1. $S_n = \sum_{i=0}^{n} X_i$, i.e. S_n is a function of $X_0, X_1, \ldots, X_n$. Note that $S_0 = X_0 = 0$.
  2. $\mathbf{E}[\,|S_n|\,] \leq \mathbf{E}\left[\,\sum_{i=0}^{n} |X_i|\,\right] \leq n < \infty$
  3. $$\begin{aligned} \mathbf{E}[\,S_{n+1}|X_0, \ldots, X_n\,] &= \mathbf{E}[\,S_n + X_{n+1}|X_0, \ldots, X_n\,] \\ &= S_n + \mathbf{E}[\,X_{n+1}|X_0, \ldots, X_n\,] = S_n \end{aligned}$$
- The usual interpretation is a Gambler who is betting 1 pound each turn, and S_n is the current profit.

The Gambler's Ruin Problem

In a more realistic situation, we start at $X_0 = K > 0$.

Let say we are going to play until we lose everything or we get $N > K$ pounds. This is no more than a random walk on the path on vertices $\{0, 1, \dots, N\}$ where vertex i is adjacent to $i + 1$.

Natural questions are

1. What is the expected reward?
2. What is the probability I leave the casino with N pounds?
3. What is the expected time I will be playing?

A more general question is: there is a good quitting strategy for the Gambler?

Today's goal is to answer those questions

A simpler exercise

The simpler quitting strategy is to quit after i rounds.

Expectation of Martingale

Let $Z_0, Z_1, \ldots,$ be a martingale w.r.t $X_0, X_1, \ldots$. Then for all $i \geq 0$, $\mathbf{E}[\, Z_i \,] = \mathbf{E}[\, Z_0 \,]$

Proof:

1. $\mathbf{E}[\, Z_{i+1} | X_0, \ldots, X_i \,] = Z_i$ because Z_i **is a martingale wrt** $X_0, X_1, \ldots$
2. $\mathbf{E}[\, Z_i \,] = \mathbf{E}[\, \mathbf{E}[\, Z_{i+1} | X_0, \ldots, X_i \,] \,] = \mathbf{E}[\, Z_{i+1} \,]$ because $\mathbf{E}[\, \mathbf{E}[\, Z | X \,] \,] = \mathbf{E}[\, Z \,]$
3. recursively, $\mathbf{E}[\, Z_{i+1} \,] = \mathbf{E}[\, Z_0 \,]$ for all $i \geq 0$

Returning to our gambling example: If we decide to stop playing after i rounds, in expectation, we finish with the same money we started

But in out quitting strategy we stop playing at a **random time** , so the previous result cannot be applied...

Martingales: Stopping Times

Not all stopping strategies are valid. We need stopping strategies that gives us stopping times.

A stopping time T w.r.t $X_0, X_1, \ldots,$ is a random variable taking values in $\{0, 1, 2, \ldots, \} \cup \{\infty\}$ such that for each $n \geq 0$:

- the event $\{T = n\}$ can be written as an event depending on $X_0, \ldots, X_n$.

The idea of a stopping time, is that we can decide to stop at time n **only** with the information we observed up to time n.

Martingales: Stopping Times

Example: Consider the Gambler's example. Are the following stopping times?

- ✓ $T = 8$.
  T is deterministic, hence it does not depend on X_i, so $\{T = n\}$ does not need to know the values of $X_{n+1}, X_{n+2}, \ldots,$
- ✓ $T =$ first time the gambler wins.
  $\{T = n\} = \{X_1 = -1, \ldots, X_{n-1} = -1, X_n = 1\}$
- ✓ $T =$ second time the gambler wins.
- ✓ $T =$ third time the gambler loses in a row.
- × First time the gambler starts a sequence of 10 loses in a row.
- ✓ First time the gambler reaches 50
- ✓ First time the gambler reaches of 50 or 0.
- × Two step before the Gambler reaches 50

Stopped Martingales

Consider a Martingale $(Z_i)_{i \geq 0}$ and a stopping time T, both with respect to $X_0, X_1, \ldots$.
Define the Stopped Martingale Z_i^T as follows

$$Z_i^T = \begin{cases} Z_i & \text{if } i \leq T \\ Z_T & \text{if } i > T \end{cases}$$

Stopped Martingales are Martingales:

- Note that $Z_i^T = Z_{i-1}^T + \mathbf{1}_{\{T \geq i\}}(Z_i - Z_{i-1})$
  - if $T \geq i$ then $Z_i^T = Z_i$, $Z_{i-1}^T = Z_{i-1}$ and $\mathbf{1}_{\{T \geq i\}} = 1$
  - if $T < i$ then $Z_i^T = Z_T$, $Z_{i-1}^T = Z_T$ and $\mathbf{1}_{\{T \geq i\}} = 0$
- Also note that $\{T \geq i\} = \bigcup_{m=0}^{i-1} \{T = m\}^c$, therefore $\{T \geq i\}$ is an event that depends at most on $X_0, \ldots, X_{i-1}$, and nothing else . Thus $\mathbf{1}_{\{T \geq i\}}$ is a function of $X_0, \ldots, X_{i-1}$.
- Now you can check the definition of martingales **(Exercise)**

Stopped Martingales

The stopped martingale is your standard martingale, but when we hit the stopping time, the martingale stops moving.

Martingales: Stopping Times

Going back to our original question: Can the gambler build a different quitting strategy that has a better outcome for him?
In general: NO.

Outline

Stopping Times

Optional Stopping Theorem

Applications to Random Walks

Optional Stopping Theorem

Optional Stopping Theorem

Let $Z_0, Z_1, \ldots,$ be a martingale w.r.t $X_0, X_1, \ldots$. Let T be a stopping time w.r.t. $X_0, \ldots$. Then $\mathbf{E}[\,Z_T\,] = \mathbf{E}[\,Z_0\,]$ whenever one of the following holds:

1. Z_i are bounded, i.e. exists $C > 0$ such that $|Z_i| \leq C$.
2. T is bounded
3. $\mathbf{E}[\,T\,] < \infty$ and there is a constant $C > 0$ such that $\mathbf{E}[\,|Z_{i+1} - Z_i||X_1, \ldots, X_i\,] < C$

The **OST** says that if at least one of those conditions holds, then $\mathbf{E}[\,Z_T\,] = \mathbf{E}[\,Z_0\,]$ where T is a stopping time.

This is equivalently to say that no matter how complex is our stopping strategy, if it is reasonable enough, then **in expectation** Z_T have the same value than Z_0

Martingales: The Gambler's Ruin Problem

Recall the problem:

- We start with K pounds. We stop playing when we reach either N pounds or 0. Of course $0 < K < N$.
- At reach round we win 1 with probability $1/2$ otherwise we lose 1.
- S_m is the amount of money we have after m rounds. $S_0 = K$
- $T = T_{0,N}$ is the stopping time defined as $T_{0,N} = \min\{i : S_i = N \text{ or } S_i = 0\}$
- We want to know S_T.

The **OST** suggests that $\mathbf{E}[\, S_T \,] = \mathbf{E}[\, S_0 \,] = K$.
Recall the conditions

OST: conditions

1. S_i are bounded, i.e. exists $C > 0$ such that $|S_i| \leq C$.
2. T is bounded, i.e. $\mathbf{P}[\, T < C \,] = 1$ for some $C > 0$
3. $\mathbf{E}[\, T \,] < \infty$ and there is a constant $C > 0$ such that $\mathbf{E}[\, |S_{i+1} - S_i||X_1, \ldots, X_i \,] < C$

OST: conditions

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- We cannot use Condition 1 because the martingale is unbounded
- We cannot use Condition 2 because in a finite amount of time there is a positive probability that something bad occurs.
- We have to use Condition 3 ! which require us to check that $\mathbf{E}[\, T \,] < \infty$!!

but we can bypass that by using the Stopped Martingale S_i^T. Recall that

$$S_i^T = \begin{cases} S_i & \text{if } i \le T \\ S_T & \text{if } i > T \end{cases}$$

by definition $S_i^T = S_i$ for $i \le T$, which is good for us because we do not care about the game after we retire

Martingales: The Gambler's Ruin Problem

The good thing: S_i^T is bounded, indeed $0 \leq S_i^T \leq N$. Then by the **OST**

$$\mathbf{E}[\, S_T \,] = \mathbf{E}\left[\, S_T^T \,\right] = \mathbf{E}\left[\, S_0^T \,\right] = \mathbf{E}[\, S_0 \,] = K$$

So in expectation, we are not doing better that not playing at all

We also get something for free. Let $P_0 = \mathbf{P}[\, S_T = 0 \,]$ and $P_N = \mathbf{P}[\, S_T = N \,]$.
Clearly,

$$P_0 + P_N = 1$$

but

$$\mathbf{E}[\, S_T \,] = N \times P_N + 0 \times P_0 = K$$

therefore $P_N = K/N$ and $P_0 = (N - K)/N$.

Remarks

- It does not matter what we do: if our strategy is such that T is a stopping time and the conditions of the OST are satisfied, then we are not going to do better
- to find better strategies we need to looking into the future (i.e. not stopping times) or breaking the OST conditions

OST: conditions

1. S_i are bounded, i.e. exists $C > 0$ such that $|S_i| \leq C$.
2. T is bounded, i.e. $\mathbf{P}[\,T < C\,] = 1$ for some $C > 0$
3. $\mathbf{E}[\,T\,] < \infty$ and there is a constant $C > 0$ such that $\mathbf{E}[\,|S_{i+1} - S_i||X_1, \ldots, X_i\,] < C$

- Breaking 1 implies to have huge amount of debt (negative values of S_i)
- We are usually breaking 2, so that is not a problem
- breaking 3 means that $\mathbf{E}[\,T\,] = \infty$, i.e. you have to be eager to play quite a lot, which also implies you need a big credit card
- in general, we break 1 and 3 at the same time

More about The Gambler's Ruin Problem

Ok, our strategy is not good.

At least: how long are we going to have fun in the casino?

- We claim that
$$Z_m = S_m^2 - m$$
is a martingale wrt $X_1, X_2, \dots$ (**Check it!!**)
- Recall $T = T_{0,N} = \min\{m : S_m = 0 \text{ or } S_m = N\}$ and that $S_0 = K$
- Condition 3 of the OST holds but unfortunately we need to check by hand that $\mathbf{E}[\,T\,] < \infty$
- the trick with the stopped martingale does not work here :(
- Anyway by the OST ,
$$K^2 - 0 = \mathbf{E}[\,Z_0\,] = \mathbf{E}[\,Z_T\,] = \mathbf{E}\Big[\,S_T^2 - T\,\Big]$$
- concluding $\mathbf{E}[\,T\,] = \mathbf{E}\big[\,S_T^2\,\big] - K^2$

- $\mathbf{E}[\,T\,] = \mathbf{E}\left[\,S_T^2\,\right] - K^2$
- Note that S_T takes two values, either 0 or N. Recall $P_0 = \mathbf{P}[\,S_T = 0\,]$ and $P_N = \mathbf{P}[\,S_T = N\,]$.
- Clearly $P_0 + P_N = 1$.
- From our previous result $P_N = K/N$
- then $\mathbf{E}[\,T\,] = \mathbf{E}\left[\,S_T^2\,\right] - K^2 = N^2 P_N - K^2 = NK - K^2 = K(N - K)$

A very good strategy

Suppose we have a huge debit card, so we can get S_n as negative as we want. A good strategy is to stop at

$$T_{good} = \min\{i : S_i = K + 1\}$$

where $S_0 = K$.

Clearly, stopping at T_{good} is a good stopping time because if we stop there we are one pound richer.

A very good strategy

For $a > 0$, consider $T_{-a,K+1}$ the following stopping time

$$T_{-a,K+1} = \min\{i : S_i = -a \text{ or } S_i = K+1\}$$

Remember that $S_0 = K$. Note that Now, note that

$$T_{-a,K+1} \leq T_{good}$$

why? Translating everything in a we have that

$$\mathbf{E}[\, T_{-a,K+1} | S_0 = K\,] = \mathbf{E}[\, T_{0,a+K+1} | S_0 = K + a\,] = (K + a)$$

Hence

$$(K + a) \leq \mathbf{E}[\, T_{good}\,]$$

but the above is valid for any $a > 0$.
Therefore

$$\mathbf{E}[\, T_{good}\,] = \infty$$

If you have time and infinitely good debit card: go ahead!

Outline

Translating the results to random walks on a line

We know the stopping time $T_{0,N}$ has expectation $K(N-K)$ when we start with K pounds. In terms of random walks this means

Hitting Times of Random Walk on a line

Consider a path on vertices $\{0, 1, \dots, N\}$ where i is connected to $i+1$. If the starting vertex is K, the expected time to reach one of the extreme points is $K(N-K)$

A few more things can be said from this

Random Walks on Graphs

Recall from John's section:

- X_t represents the position after step t of a the random walk on a graph G. X_0 is the initial vertex.
- τ_y is the number of steps it takes to reach vertex y
- $h_{x,y}$ is the expected number of steps it takes to reach vertex y starting from x, i.e. $\mathbf{E}[\,\tau_y | X_0 = x\,]$
- the cover time is the expected time to visit all the vertices of the graph starting from the worst position.

A few problems

- Consider out path on vertices $\{0, \dots, N\}$, and suppose $X_0 = K$. Compute $h_{K,N}$
- Compute the cover time of a path on $\{0, \dots, N\}$ when N is even. What about when N is odd?
- Consider a cycle on vertices $\{0, 1, \dots, N\}$ where vertex i is adjacent to $i + 1$, and 0 is adjacent to N. Compute the cover time.
- Consider a cycle on vertices $\{0, 1, \dots, N\}$. Define T by

  $$T = \min\{m : \cup_{i=0}^{m} X_i = \{0, \dots, N\}\}$$

  T is the first time all vertices has been covered. Compute $\mathbf{P}[X_T = i | X_0 = 0]$ for $i \in \{1, \dots, N\}$.