Lecture 6: Concentration Inequalities - Introduction to Martingales

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Outline

More Chernoff Bounds

Conditional Expectation

Chernoff Bounds

Remember the Chernoff Bounds from the previous lecture..

Chernoff Bounds: upper tails

Suppose $X_1, \ldots, X_n$ are independent Bernoulli random variables with parameter p_i. Let $X = X_1 + \ldots + X_n$ and $\mu = \mathbf{E}[X] = \sum p_i$. Then, for any $\delta > 0$ it holds that

$$\mathbf{P}[X \geq (1+\delta)\mu] \leq \left[\frac{e^{\delta}}{(1+\delta)^{(1+\delta)}}\right]^{\mu}.$$

and for $t > \mu$ it holds that

$$\mathbf{P}[X \geq t] \leq e^{-\mu}\left(\frac{e\mu}{t}\right)^{t},$$

Chernoff Bounds

.. and the lower tails..

Chernoff Bounds: Lower Tails

Suppose $X_1, \ldots, X_n$ are independent Bernoulli random variables with parameter p_i. Let $X = X_1 + \ldots + X_n$ and $\mu = \mathbf{E}[\, X \,] = \sum p_i$. Then, for any $\delta > 0$ it holds that

$$\mathbf{P}[\, X \leq (1-\delta)\mu \,] \leq \left[\frac{e^{-\delta}}{(1-\delta)^{1-\delta}} \right]^{\mu}.$$

and for any $t < \mu$

$$\mathbf{P}[\, X \leq t \,] \leq e^{-\mu} \left(\frac{e\mu}{t} \right)^{t}.$$

Chernoff Bounds

..and the nicer version!

Nicer Chernoff Bounds

Suppose $X_1, \dots, X_n$ are independent Bernoulli random variables with parameter p_i. Let $X = X_1 + \dots + X_n$ and $\mu = \mathbf{E}[\,X\,] = \sum p_i$. Then,

- For all $t > 0$,

$$\mathbf{P}[\,X \geq \mathbf{E}[\,X\,] + t\,] \leq e^{-2t^2/n}$$

$$\mathbf{P}[\,X \leq \mathbf{E}[\,X\,] - t\,] \leq e^{-2t^2/n}$$

- For $0 < \delta < 1$,

$$\mathbf{P}[\,X \geq (1+\delta)\mathbf{E}[\,X\,]\,] \leq \exp\left(-\frac{\delta^2 \mathbf{E}[\,X\,]}{3}\right)$$

$$\mathbf{P}[\,X \leq (1-\delta)\mathbf{E}[\,X\,]\,] \leq \exp\left(-\frac{\delta^2 \mathbf{E}[\,X\,]}{2}\right)$$

Chernoff Bound: Extension to other Random Variables

- Most of the time we will use Chernoff Bounds for sum of independent Bernoulli random variables
- but not always
- it does not hurt to know how to derive similar bounds for other random variables

Remember the key steps:

Chernoff Bound recipe

1. Let $\lambda > 0$, then

$$\mathbf{P}[\, X \geq (1+\delta)\mu \,] \leq e^{-\lambda(1+\delta)\mu} \mathbf{E}\left[\, e^{\lambda X} \,\right]$$

2. Compute an upper bound for $\mathbf{E}\left[\, e^{\lambda X} \,\right]$
3. Optimise the value of $\lambda > 0$.

Exercise:

- Let X be a Poisson random variable of mean μ. Prove that

$$\mathbf{E}\left[e^{\lambda X} \right] = e^{\mu(e^{\lambda}-1)}$$

and deduce that for $t \geq \mu$

$$\mathbf{P}[\, X \geq t \,] \leq e^{-\mu} \left(\frac{e\lambda}{t} \right)^{t} \quad \text{and} \quad \mathbf{P}[\, X \geq (1+\delta)\mu \,] \leq e^{-\delta^2 \mu},$$

and the corresponding lower tails.

- Let X be a Normal random variable of mean μ and variance σ^2. Prove that

$$\mathbf{E}\left[e^{\lambda X} \right] = e^{\mu\lambda + \sigma^2\lambda^2/2},$$

and deduce that for $t > \mu$

$$\mathbf{P}[\, X \geq t \,] \leq e^{-(t-\mu)^2/2}.$$

Hoeffding's Extension

- Beside sums of independent Bernoulli Random variables, sums of independent and bounded random variables is very important in applications.
- Unfortunately the distribution of the X_i will be unknown or very hard to compute, thus it will be very hard to compute the moment-generating function of X_i.
- Hoeffding's Lemma helps us here

You can always consider $X' = X - \mathbf{E}[X]$

Hoeffding's Extension Lemma

Let X be a random variable with mean 0 such that $a \leq X \leq b$, then for all $\lambda \in \mathbb{R}$.

$$\mathbf{E}\left[e^{\lambda X}\right] \leq \exp\left(\frac{(b-a)^2\lambda^2}{8}\right)$$

Chernoff-Hoeffding Bounds

Chernoff-Hoeffding's Bounds

Let $X_1, \ldots, X_n$ be independent random variable with mean μ_i such that $a_i \leq X_i \leq b_i$. Let $X = X_1 + \ldots + X_n$, and let $\mu = \mathbf{E}[\, X\,] = \sum_{i=1}^{n} \mu_i$. Then for any $t > 0$

$$\mathbf{P}[\, X \geq \mu + t\,] \leq \exp\left[\frac{-2t^2}{\sum_{i=1}^{n}(b_i - a_i)^2}\right]$$

and

$$\mathbf{P}[\, X \leq \mu - t\,] \leq \exp\left[\frac{-2t^2}{\sum_{i=1}^{n}(b_i - a_i)^2}\right]$$

Proof:

- Let $X_i' = X_i - \mu_i$ and $X' = X_1' + \ldots, X_n'$, then $\mathbf{P}[\, X \geq \mu + t\,] = \mathbf{P}[\, X' \geq t\,]$
- $\mathbf{P}[\, X' \geq t\,] \leq e^{-\lambda t} \prod_{i=1}^{n} \mathbf{E}\left[\, e^{\lambda X_i'}\,\right] \leq \exp\left[-\lambda t + \frac{\lambda^2}{8} \sum_{i=1}^{n}(b_i - a_i)^2\right]$
- Choose $\lambda = \frac{4t}{\sum_{i=1}^{n}(b_i - a_i)^2}$ to get the result.

This is not magic! you just need to optimise on λ

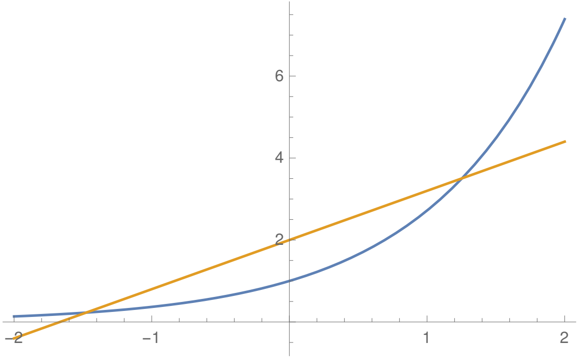
Hoeffding's Extension Lemma

Let X be a random variable with mean 0 such that $a \leq X \leq b$, then for all $\lambda \in \mathbb{R}$.

$$\mathbf{E}\left[e^{\lambda X}\right] \leq \exp\left(\frac{(b-a)^2\lambda^2}{8}\right)$$

Proof (for $\lambda \geq 0$):

- $f(x) = e^{\lambda x}$ is a convex function.



Hoeffding's Extension Lemma

Let X be a random variable with mean 0 such that $a \leq X \leq b$, then for all $\lambda \in \mathbb{R}$.

$$\mathbf{E}\left[e^{\lambda X}\right] \leq \exp\left(\frac{(b-a)^2\lambda^2}{8}\right)$$

Proof (for $\lambda \geq 0$):

1. $f(x) = e^{\lambda x}$ is a convex function.
2. As $a \leq X \leq b$, we consider the points $(a, e^{\lambda a})$ and $(b, e^{\lambda b})$
3. The straight line between those points is always above the graph of $e^{\lambda x}$
4. i.e.

$$e^{\lambda X} \leq \frac{b-X}{b-a}e^{\lambda a} + \frac{X-a}{b-a}e^{\lambda b}$$

5. Then

$$\mathbf{E}\left[e^{\lambda X}\right] \leq \frac{be^{\lambda a}}{b-a} - \frac{ae^{\lambda b}}{b-a}$$

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5. Then
$$\mathbf{E}\left[e^{\lambda X} \right] \leq \frac{be^{\lambda a}}{b-a} - \frac{ae^{\lambda b}}{b-a}$$
6. Consider
$$\phi(\lambda) = \log\left(\frac{be^{\lambda a}}{b-a} - \frac{ae^{\lambda b}}{b-a} \right)$$
and check that (Exercise)
    - $\phi(0) = 0$
    - $\phi'(0) = 0$
    - $\phi''(t) \leq (b-a)^2/4$ for all $t \in \mathbb{R}$
7. For $t \geq 0$, use that $\phi'(t) = \int_0^t \phi''(x)dx \leq t(b-a)^2/4$
8. For $t \geq 0$, use that
$$\phi(t) = \int_0^t \phi'(x)dx \leq \int_0^t x(b-a)^2/4dx \leq t^2(b-a)^2/8$$
9. replace $t = \lambda$ for non-negative λ.

Chernoff-Bounds: Final Remarks

- There are several version of Chernoff-style Bounds that work for sum of independent random variables.
- The proof of all of them usually follows the same **recipe**
- Some bounds include more information about the random variables, e.g. the variance
- the limit is the amount of information we have about the random variables and our ability to manipulate/bound quantities.

Beyond sum of independent variables

Can we prove concentration of other type of random variables? Yes.. but

- There is no general tool to prove concentration beyond the basic **recipe**
- but in general it is very hard to compute moment generating functions
- It is worth trying to transform the problem into the setting of sum of independent random variable
- If everything fails.. There are a few other families of random variables for which proving concentration is doable One of them are the so-called **Martingales**

Outline

More Chernoff Bounds

Conditional Expectation

Conditional Expectation

Before talking about martingales, we need to review **conditional expectation.**

- Given two events A and B with $\mathbf{P}[\,A\,] > 0$ we define $\mathbf{P}[\,B|A\,] = \mathbf{P}[\,B \cap A\,]\,/\mathbf{P}[\,A\,]$.
- if $\mathbf{P}[\,A\,] = 0$, the usual convention is that $\mathbf{P}[\,B|A\,] = 0$.
- Given a discrete random variable Y, we define its conditional expectation with respect to the event A by

$$\mathbf{E}[\,Y|A\,] = \sum_b b\mathbf{P}[\,Y = b|A\,]$$

- a particular case is when the event $A = \{X = a\}$ where X is another discrete random variable. In such a case we define the function $f(a)$ by

$$f(a) = \mathbf{E}[\,Y|X = a\,]\,,$$

- We define the conditional expectation $\mathbf{E}[\,Y|X\,]$, as the **random variable** that takes the value $\mathbf{E}[\,Y|X = a\,]$ then $X = a$, i.e. $f(X)$.

Important Remarks

- The conditional expectation of Y w.r.t a event A, $\mathbf{E}[Y|A]$ is a deterministic number .
- The conditional expectation of Y w.r.t a random variable X, $\mathbf{E}[Y|X]$ is a random variable .
- X can be a random vector $(X_1, \ldots, X_N)$ in the definition of $\mathbf{E}[Y|X]$.
- There is a definition of conditional expectation with respect to general random variables[1], but most of the results in the discrete setting extend to the continuous setting.
- The conditional expectation $\mathbf{E}[Y|X]$ is always a function of X.
- Behind conditional expectation there is the notion of information [2]. The standard notion of expectation is like 'the best estimate of a random variable given no information of it', while the conditional expectation given X is like 'the best estimate of a random variable given the information of X'

[1] such a definition require the understanding of Measure theory
[2] Measure theory, again

Conditional Expectation: two dices

Suppose we independently roll two standard 6-sided dice. Let X_1 and X_2 the observed number in the first and second dice respectively. We compute a few conditional expectations.

1. $\mathbf{E}[X_1 + X_2 | X_1] = 3.5 + X_1$. Why? Because if $X_1 = a$ then

$$\begin{aligned}
\mathbf{E}[X_1 + X_2 | X_1 = a] &= \sum_{b=1}^{12} b\mathbf{P}[X_1 + X_2 = b | X_1 = a] \\
&= \sum_{b=1}^{12} b\mathbf{P}[X_1 + X_2 = b, X_1 = a] / \mathbf{P}[X_1 = a] \\
&= \sum_{b=1}^{12} b\mathbf{P}[X_2 = b - a, X_1 = a] / \mathbf{P}[X_1 = a] \\
\color{red}{X_1 \text{ indep } X_2} \quad &= \sum_{b=1}^{12} b\mathbf{P}[X_2 = b - a] \\
&= \sum_{c=1}^{6} (c + a)\mathbf{P}[X_2 = c] \\
&= 3.5 + a
\end{aligned}$$

Conditional Expectation: Properties

1. $\mathbf{E}[\,\mathbf{E}[\,Y|X\,]\,] = \mathbf{E}[\,Y\,]$.
2. $\mathbf{E}[\,1|X\,] = 1$
3. **Linearity :**
   - For any constant $c \in \mathbb{R}$, $\mathbf{E}[\,cY|X\,] = c\mathbf{E}[\,Y|X\,]$
   - $\mathbf{E}[\,Y + Z|X\,] = \mathbf{E}[\,Y|X\,] + \mathbf{E}[\,Z|X\,]$
4. If X is independent of Y, then $\mathbf{E}[\,Y|X\,] = \mathbf{E}[\,Y\,]$.
5. if Y is a function of X[3], i.e. $Y = f(X)$, then $\mathbf{E}[\,YZ|X\,] = Y\mathbf{E}[\,Z|X\,]$. Particularly, $\mathbf{E}[\,X|X\,] = X$
6. **Tower Property:**
   - $\mathbf{E}[\,\mathbf{E}[\,X|(Z,Y)\,]\,|Y\,] = \mathbf{E}[\,X|Y\,]$.
7. **Jensen Inequality:**
   - if f is a convex real function, then $f(\mathbf{E}[\,X|Y\,]) \leq \mathbf{E}[\,f(X)|Y\,]$.

By using this properties, everything becomes a bit easier, e.g., our two dices example

$$\mathbf{E}[\,X_1 + X_2|X_1\,] \overset{p3}{=} \mathbf{E}[\,X_1|X_1\,] + \mathbf{E}[\,X_2|X_1\,] \overset{p5,p4}{=} X_1 + \mathbf{E}[\,X_2\,] = X_1 + 3.5$$

[3] measurable function

Exercise: Prove the properties

1. $\mathbf{E}[\,\mathbf{E}[\,Y|X\,]\,] = \mathbf{E}[\,Y\,]$.
   Proof: e.g.

$$\sum_x \mathbf{E}[\,Y|X=x\,]\,\mathbf{P}[\,X=x\,] = \sum_x \sum_y y\mathbf{P}[\,Y=y|X=x\,]\,\mathbf{P}[\,X=x\,]$$
$$= \sum_x \sum_y y\mathbf{P}[\,Y=y, X=x\,]$$
$$= \sum_y y\mathbf{P}[\,Y=y\,]$$

Example: Expectation of a Geometric Random Variable

Suppose $X_1, X_2, \ldots,$ are an infinite sequence of independent Bernoulli (coins) random variables of parameter p, i.e. $\mathbf{P}[\, X_i = 1\,] = p$. Define

$G = \min\{k \geq 1 : X_k = 1\}$, which is the number of coins we have to observe until we get a head.

G has geometric distribution of parameter p. Indeed $\mathbf{P}[\, G = k\,] = p(1-p)^{k-1}$.

The expectation of G is given by the formula

$$\mathbf{E}[\, G\,] = \sum_{k=1}^{\infty} kp(1-p)^{k-1}$$

Let say that we forgot how to compute that type of sums...

We can compute $\mathbf{E}[\,G\,]$ by other means.

- $\mathbf{E}[\,G\,] \overset{p1}{=} \mathbf{E}[\,\mathbf{E}[\,G|X_1\,]\,]$
- $G = X_1 + (1 - X_1)(1 + G')$ where G' is the number of coins we need to wait to see a head after the first coin.
- $\mathbf{E}[\,X_1 + (1 - X_1)(1 + G')|X_1\,] \overset{p3,p5}{=} X_1 + (1 - X_1)\mathbf{E}[\,1 + G'|X_1\,]$
- G' has geometric distribution of parameter p and it is independent of X_1. Hence
$$\mathbf{E}\left[\,1 + G'|X_1\,\right] \overset{p4}{=} \mathbf{E}\left[\,1 + G'\,\right] = 1 + \mathbf{E}[\,G\,]$$
- Solve
$$\mathbf{E}[\,G\,] = p + (1 - p)(1 + \mathbf{E}[\,G\,])$$

Example: Balls into Bins

Suppose we have n bins but a random number of balls, say M. Suppose M has finite expectation. What is the expected number of balls in the first bin?.

1. Recall balls are assigned to bins uniformly at random and independent of everything
2. Let $X_i = 1$ if the ball i falls in bin 1
3. The total number of balls in bin 1 is $\sum_{i=1}^{M} X_i$ (recall M is a random variable, and M is independent of X_i)

4. $\mathbf{E}\left[\sum_{i=1}^{M} X_i\right] \overset{p1}{=} \mathbf{E}\left[\mathbf{E}\left[\sum_{i=1}^{M} X_i \,\middle|\, M\right]\right]$

5. $\mathbf{E}\left[\sum_{i=1}^{M} X_i \,\middle|\, M\right] = \mathbf{E}\left[\sum_{i=1}^{\infty} X_i \mathbf{1}_{\{i \leq M\}} \,\middle|\, M\right] \overset{p3}{=}{}^{4} \sum_{i=1}^{\infty} \mathbf{E}\left[X_i \mathbf{1}_{\{i \leq M\}} \,\middle|\, M\right]$

6. $\mathbf{E}\left[X_i \mathbf{1}_{\{i \leq M\}} \,\middle|\, M\right] \overset{p5}{=} \mathbf{1}_{\{i \leq M\}} \mathbf{E}\left[X_i \,\middle|\, M\right] \overset{p4}{=} \mathbf{1}_{\{i \leq M\}} \mathbf{E}[X_i] = \mathbf{1}_{\{i \leq M\}} \cdot (1/n)$

7. Replacing 6 in 5: $\mathbf{E}\left[\sum_{i=1}^{M} X_i \,\middle|\, M\right] = \sum_{i=1}^{\infty} (1/n) \cdot \mathbf{1}_{\{i \leq M\}}$

8. replacing 7 into 4: $\mathbf{E}\left[\sum_{i=1}^{M} X_i\right] = \sum_{i=1}^{\infty} (1/n) \cdot \mathbf{P}[i \leq M] =^{5} (1/n) \cdot \mathbf{E}[M]$

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[4] Technically linearity works for a finite sum, but in most cases it can be done for infinite case. We need measure theory to justify that

[5] See Q2 of the Homework Assessment