Outline

Concentration Inequalities

Chernoff Bounds

Balls into Bins

Proof of Chernoff Bounds

Randomised QuickSort
Concentration Inequalities

- **Concentration** refers to the phenomena where random variables are very close to their mean.
- This is very useful in randomised algorithms as it ensures an almost deterministic behaviour.
- It gives us the best of two worlds:
  1. **Randomised Algorithms:** Easy to Design and Implement
  2. **Deterministic Algorithms:** They do what they claim to do
Recall the Markov and Chebyshev inequalities from the first lecture.

**Markov Inequality**

If $X$ is a non-negative random variable and $a > 0$, then

$$P[X \geq a] \leq \frac{E[X]}{a}.$$

**Chebyshev Inequality**

If $X$ is a random variable and $a > 0$, then

$$P[|X - E[X]| \geq a] \leq \frac{\text{Var}[X]}{a^2}.$$

Let $f : \mathbb{R} \to [0, \infty)$ and increasing, then $Y = f(X) \geq 0$, and thus

$$P[X \geq a] \leq P[f(X) \geq f(a)] \leq \frac{E[f(X)]}{f(a)}.$$

Similarly, if $g : \mathbb{R} \to [0, \infty)$ and decreasing, then $Y = g(X) \geq 0$, and thus

$$P[X \leq a] \leq P[g(X) \geq g(a)] \leq \frac{E[g(X)]}{g(a)}.$$

By choosing an appropriate function we can obtain inequalities that are much sharper than the Markov and Chebyshev Inequality.
Example: coin flip

Consider $n$ fair coins and let $X$ be the total number of head. In an experiment we expect to see around $n/2$ heads. Can we justify that? Let $\delta > 0$

- Markov inequality:

$$P[X \geq (1 + \delta)(n/2)] \leq \frac{E[X]}{(1 + \delta)(n/2)} = \frac{n/2}{(1 + \delta)(n/2)} = \frac{1}{1 + \delta}$$

- Chebychev inequality:

$$P[(X - n/2) \geq \delta(n/2)] \leq P[(X - n/2)^2 \geq (\delta(n/2))^2]$$

$$\leq \frac{4\text{Var}[X]}{\delta^2 n^2} = \frac{1}{\delta^2 n}$$

Not good! Independent of $n$

Better! Linear in $n$
Markov and Chebychev use the first and second moment of the random variable. Can we keep going?

- Yes.

We can consider first, second, third and more moments! that is the basic idea behind the Chernoff Bounds.
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Suppose $X_1, \ldots, X_n$ are independent Bernoulli random variables with parameter $p_i$. Let $X = X_1 + \ldots + X_n$ and $\mu = \mathbb{E}[X] = \sum p_i$. Then, for any $\delta > 0$ it holds that

$$
P[X \geq (1 + \delta)\mu] \leq \left[ \frac{e^{\delta}}{(1 + \delta)^{1+\delta}} \right]^{\mu}.
$$

and for $t > \mu$ it holds that

$$
P[X \geq t] \leq e^{-\mu} \left( \frac{e^{\mu}}{t} \right)^t,
$$
Consider $n$ fair coins and let $X$ be the total number of head. Then

$$P[X \geq (1 + \delta)(n/2)] \leq \left[ \frac{e^\delta}{(1 + \delta)^{1+\delta}} \right]^{n/2}$$

Node that the above expression equals 1 only for $\delta = 0$, and then it gives a value strictly less than 1 (Ex: check this!). Note as well the inequality is exponential in $n$, (for fixed $\delta$) i.e. much better than Chebychev inequality.
Example: Coin Flip

Consider 100 independent coin flips. We wish to find an upper bound on the probability that the number of heads is greater or equal than 75.

- **Markov’s inequality**: \( X = \sum_{i=1}^{100} X_i, \; X_i \in \{0, 1\} \) and \( E[X] = 100 \cdot \frac{1}{2} = 50 \).

\[
P[X \geq 3/2 \cdot E[X]] \leq \frac{2}{3} = 0.666.
\]

- **Chebyshev’s inequality**: \( \text{Var}[X] = \sum_{i=1}^{100} \text{Var}[X_i] = 100 \cdot (1/2)^2 = 25 \).

\[
P[|X - \mu| \geq t] \leq \frac{\text{Var}[X]}{t^2},
\]

and plugging in \( t = 25 \) gives an upper bound of \( 25/25^2 = 1/25 = 0.04 \), much better than what we obtained by Markov’s inequality.

- **The Chernoff bound (first)** with \( \delta = 1/2 \) gives:

\[
P[X \geq 3/2 \cdot E[X]] \leq \left( \frac{e^{1/2}}{(3/2)^{3/2}} \right)^{50} = 0.004472
\]

- the exact probability is 0.00000028 \ldots

**Chernoff bound yields a more accurate result but needs independence!**
Histogram for number of heads

X
Density
30 40 50 60 70
0.00 0.02 0.04 0.06 0.08
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Randomised QuickSort
Example: Balls into Bins

Recall the balls into bins process: Assign \( m \) balls uniformly and independently to \( n \) bins.

Balls into Bins question

How large is the maximum load?

- Focus in one bin. Let \( X_i \) the indicator variable that indicates if ball \( i \) is assigned to this bin. The total balls in the bin is given by \( X = \sum_i X_i \). Note that \( p_i = P[X_i = 1] = 1/n \).
- Suppose that \( m = 2n \log n \), then \( \mu = E[X] = 2 \log n \)

By the Chernoff Bound,

\[
P[X \geq t] \leq e^{-\mu} \left( \frac{e \mu}{t} \right)^t
\]

\[
P[X \geq 6 \log n] \leq e^{-2 \log n} \left( \frac{2e \log n}{6 \log n} \right)^{6n \log n} \leq e^{-2 \log n} = n^{-2}
\]
Example: Balls into Bins

- Let $E_j$ be the event that bin $j$ receives more than $6 \log n$ balls.
- We are interested in the probability that at least one bin receives more than $6 \log n$ balls.
- This is the event $\bigcup_{j=1}^{n} E_j$.
- By the Union Bound, $P[\bigcup_{j=1}^{n} E_j] \leq \sum_{j=1}^{n} P[E_j] \leq n \times n^{-2} = n^{-1}$.
- Therefore $whp$, no bin receives more than $6 \log n$ balls.
- Note that the max loaded bin receives at least $2 \log n$. So our bound is pretty sharp.

**whp** stands for *with high probability*:
An event $\mathcal{E}$ (that implicitly depends on an input parameter $n$) occurs $whp$ if $P[\mathcal{E}] \to 1$ as $n \to \infty$.
This is a very standard notation in randomised algorithms but it may vary from author to author. Be careful!
Example: Balls into Bins

Consider now the case \( m = n \), i.e. same number of balls and bins. Using the Chernoff Bounds

\[
\Pr[ X > t ] \leq e^{-1} \left( \frac{e}{t} \right)^t \leq \left( \frac{e}{t} \right)^t
\]

\[
\Pr[ X \geq t ] \leq e^{-\mu} \left( \frac{e\mu}{t} \right)^t
\]

By setting \( t = 4 \log n / \log \log n \), we obtain that \( \Pr[ X > t ] \leq n^{-2} \). Indeed

\[
\left( \frac{e \log \log n}{4 \log n} \right)^{4 \log n / \log \log n} = \exp \left( \frac{4 \log n}{\log \log n} \cdot \log \left( \frac{e \log \log n}{4 \log n} \right) \right)
\]

(1)

The term inside the exponential is

\[
\left( \frac{4 \log n}{\log \log n} \cdot (\log(4/e) + \log \log \log n - \log \log n) \right) \leq \left( \frac{4 \log n}{\log \log n} \left( -\frac{1}{2} \log \log n \right) \right)
\]

obtaining that \( \Pr[ X > t ] \leq n^{-4/2} = n^{-2} \).

This inequality only works for large enough \( n \).
Example: Balls into Bins

We just proved that

$$\Pr[X > 4 \log n / \log \log n] \leq n^{-2},$$

thus by the union bound, no bin receives more than $O(\log n / \log \log n)$ balls with probability at least $1 - 1/n$.

- You will see in your Exercise class how to prove that whp at least one bin receives at least $c \log n / \log \log n$ balls, for some $c$. You will need to use the Probabilistic Method mentioned in the first lecture.
Conclusions

- If the number of balls is $2 \log n$ times $n$ (the number of bins), then to distribute balls at random is a good algorithm.
  - This is because the worst case load is whp. $6 \log n$, while the expected number of balls is $2 \log n$.

- For the case $n = m$, the algorithm is not good, because the maximum load is whp. $\Theta(\log n / \log \log n)$, while the expected number of balls per bin is 1.

- For the case $n = m$, we can improve the balls into bin process by sampling two bins in each step, then assigning the ball into the bin with lesser load. This gives a maximum load $\Theta(\log \log n)$ with probability at least $1 - 1/n$.

This is called the power of two choices: It is a standard technique to improve the performance of randomised algorithms.
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Proof of Chernoff Bounds

Randomised QuickSort
Chernoff Bound: Proof

Suppose $X_1, \ldots, X_n$ are independent Bernoulli random variables with parameter $p_i$. Let $X = X_1 + \ldots + X_n$ and $\mu = \mathbb{E}[X] = \sum p_i$. Then, for any $\delta > 0$ it holds that

$$
\mathbb{P}[X \geq (1 + \delta)\mu] \leq \left[ \frac{e^\delta}{(1 + \delta)^2} \right]^\mu.
$$

Proof:

1. For $\lambda > 0$,

$$
\mathbb{P}[X \geq (1 + \delta)\mu] \leq \mathbb{P}[e^{\lambda X} \geq e^{\lambda(1+\delta)\mu}] \leq e^{-\lambda(1+\delta)\mu} \mathbb{E}[e^{\lambda X}]
$$

2. $\mathbb{E}[e^{\lambda X}] = \mathbb{E}[e^{\lambda \sum X_i}] = \prod_{i=1}^n \mathbb{E}[e^{\lambda X_i}]$

3. $\mathbb{E}[e^{\lambda X_i}] = e^{\lambda p_i} + (1 - p_i) = 1 + p_i(e^\lambda - 1) \leq e^{p_i(e^\lambda - 1)}$

for $1 + x \leq e^x$ for $x > 0$
Chernoff Bound: Proof

1. For $\lambda > 0$,
   
   $$
P[ X \geq (1 + \delta)\mu ] = \mathbb{P}[ e^{\lambda X} \geq e^{\lambda(1+\delta)\mu} ] \leq e^{-\lambda(1+\delta)\mu} \mathbb{E}[ e^{\lambda X} ]
   $$

   $e^{\lambda X}$ is incr

2. $\mathbb{E}[ e^{\lambda X} ] = \mathbb{E}[ e^{\lambda \sum X_i} ] = \prod_{i=1}^{n} \mathbb{E}[ e^{\lambda X_i} ]$

3. $\mathbb{E}[ e^{\lambda X_i} ] = e^{\lambda p_i} + (1 - p_i) = 1 + p_i(e^{\lambda} - 1) \leq e^{p_i(e^{\lambda} - 1)}$

   $1 + x \leq e^x$
   
   for $x > 0$

4. Putting all together

   $$
P[ X \geq (1 + \delta)\mu ] \leq e^{-\lambda(1+\delta)\mu} \prod_{i=1}^{n} e^{p_i(e^{\lambda} - 1)} = e^{-\lambda(1+\delta)\mu} e^{\mu(e^{\lambda} - 1)}
   $$

5. Choose $\lambda = \log(1 + \delta) > 0$ to get the result.
Chernoff Bound: The recipe

The proof of the Chernoff bound is based in three key steps. These are

1. Let $\lambda > 0$, then

$$ P[ X \geq (1 + \delta)\mu ] \leq e^{-\lambda(1+\delta)\mu} E[ e^{\lambda X} ] $$

2. Compute an upper bound for $E[ e^{\lambda X} ]$ (*This is the hard one*)

3. Optimise the value of $\lambda > 0$.

The function $\lambda \to E[ e^{\lambda X} ]$ is called the moment-generating function of $X$ and it is very important to obtain sharp concentration inequalities.

**Exercise:** prove that $P[ X \geq t ] \leq e^{-\mu} \left( \frac{e\mu}{t} \right)^t$, 


We can also use Chernoff Bounds to show that a random variable is not too small compared to its mean.

Suppose $X_1, \ldots, X_n$ are independent Bernoulli random variables with parameter $p_i$. Let $X = X_1 + \ldots + X_n$ and $\mu = E[X] = \sum p_i$. Then, for any $\delta > 0$ it holds that

$$P[X \leq (1 - \delta)\mu] \leq \left[ \frac{e^{-\delta}}{(1 - \delta)^{1-\delta}} \right]^\mu.$$  

and for any $t < \mu$

$$P[X \leq t] \leq e^{-\mu} \left( \frac{e^{\mu}}{t} \right)^t.$$  

Exercise: Prove it.  

Hint: multiply both sides by $-1$ and repeat the proof of the Chernoff Bound.
**The useful Chernoff Bounds**

Our current form of the Chernoff Bound is rather atrocious. We can derive a slightly weaker but more readable version.

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### Nicer Chernoff Bounds

Suppose $X_1, \ldots, X_n$ are independent Bernoulli random variables with parameter $p_i$. Let $X = X_1 + \ldots + X_n$ and $\mu = E[X] = \sum p_i$. Then,

- For all $t > 0$,
  \[
  P[X \geq EX + t] \leq e^{-2t^2/n}
  \]
  \[
  P[X \geq EX - t] \leq e^{-2t^2/n}
  \]

- For $0 < \delta < 1$,
  \[
  P[X \geq (1 + \delta)E[X]] \leq \exp \left( -\frac{\delta^2E[X]}{3} \right)
  \]
  \[
  P[X \leq (1 - \delta)E[X]] \leq \exp \left( -\frac{\delta^2E[X]}{2} \right)
  \]

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**Exercise: Prove it**
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Randomised QuickSort
Applications: QuickSort

Quick sort is a sorting algorithm that works as following.

**Algorithm: QuickSort**

- **Input:** Array of different number $A$.
- **Output:** array $A$ sorted in increasing order
  - Pick an element from the array, the so-called pivot.
  - If $|A| = 0$ or $|A| = 1$; return $A$.
  - Else
    - Generate two subarrays $A_1$ and $A_2$:
      - $A_1$ contains the elements that are smaller than the pivot.
      - $A_2$ contains the elements that are greater than the pivot.
    - Recursively sort $A_1$ and $A_2$.

E.g. Let $A = (2, 8, 9, 1, 7, 5, 6, 3, 4)$, choose 6 as pivot, then we get $A_1 = (2, 1, 5, 3, 4)$ and $A_2 = (8, 9, 7)$.

It is well-known that the worst-case complexity (number of comparisons) of quick sort is $O(n^2)$. This happens when pivots are pretty bad, generating one large array and one small array.
Applications: QuickSort

Note that the number of comparison performed in quick sort is equivalent to the sum of the height of all nodes in the tree. In this case

$$0 + 1 + 1 + 2 + 2 + 2 + 3 + 3 + 3 = 17.$$
How to pick a **good pivot**? we don’t, **just pick one at random.**

Let’s analyse quicksort with random pivots.

1. Consider $n$ different number, wlog, $\{1, \ldots, n\}$
2. let $H_i$ be the last level where $i$ appears in the tree. Then the number of comparison is $H = \sum_{i=1}^{n} H_i$
3. we will prove that exists $C > 0$ such that

$$P[\forall i, H_i \leq C \log n] \geq 1 - 1/n$$

4. actually, we will prove something equivalent but easier: we will prove that all leaves of the tree are at distance at most $C \log n$ from the root with probability at least $1 - 1/n$.

5. then $H = \sum_{i=1}^{n} H_i \leq Cn \log n$, with probability at least $1 - 1/n$.
Applications: QuickSort

- Let \( P \) be a path from the root to a leaf. A node in \( P \) is called **good** if the corresponding pivot partition the array into two subarrays each of size at least \( 1/3 \) of the previous one, the node is **bad** otherwise.
- Denote by \( s_t \) the size of the array at level \( t \) in \( P \).

E.g. Path: \((2, 8, 9, 1, 7, 5, 6, 3, 4) \rightarrow (2, 1, 5, 3, 4) \rightarrow (5, 3, 4) \rightarrow (5)\)

The vertices are: good, bad, good

\( s_0 = 9 \), \( s_1 = 5 \), \( s_2 = 3 \), \( s_3 = 1 \).
Let $P$ be a path from the root to a leaf. A node in $P$ is called **good** if the corresponding pivot partition the array into two subarrays each of size at least $1/3$ of the previous one, the node is **bad** otherwise.

Denote by $s_t$ the size of the array at level $t$ in $P$.

After a good vertex we have that $s_t \leq (2/3)s_{t-1}$.

Therefore, there are at most $T = \frac{\log n}{\log(3/2)} \leq 2 \log n$ good nodes in a path $P$.

Set $C = 21$ and suppose that $|P| > C \log n$.

this implies that the number of bad vertices in the first $21 \log n$ nodes is more than $19 \log n$. 
- Consider the first \( \lfloor 21 \log n \rfloor \) vertices of \( P \). Denote by \( X_i = 1 \) if the node at height \( i \) of \( P \) is bad, and \( X_i = 0 \) if it is good. Let \( X = \sum_{i=1}^{\lfloor 21 \log n \rfloor} X_i \).

- Note that the \( X_i \)'s are independent and \( P[ X_i = 1 ] = 2/3 \), and \( E[ X ] = (2/3)21 \log n = 14 \log n \). Then, by the (nicer) Chernoff Bounds

\[
P[ X > E[ X ] + t ] \leq e^{-2t^2/n}
\]

\[
P[ X > 19 \log n ] = P[ X > E[ X ] + 5 \log n ] \leq e^{-2(5 \log n)^2/(21 \log n)} = e^{-(50/21) \log n} \leq 1/n^2.
\]

- Hence, we conclude the path has more than \( 21 \log n \) nodes with probability at most \( n^{-2} \). There are at most \( n \) leaves, then by union bound, the probability that at least one path has more than \( 21 \log n \) nodes is \( n^{-1} \).
Remarks

- It is known that no sorting algorithm based on comparison takes less than $\Omega(n \log n)$
- The constant $C$ can be improved a little bit, but in any case we will obtain that our randomised version of QuickSort that whp compares $O(n \log n)$ pairs
- It is possible to deterministically choose the best pivot that divide the array into two subarrays of the same size.
- The later requires to compute the median of the array in linear time, which is not easy to do
- Randomised solution for QuickSort is much easier to implement.