Lecture 4: Card Shuffling and Covertime

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Lent 2019

UNIVERSITY OF CAMBRIDGE

Outline

Shuffling and Strong Stationary Times

Covertime

$s - t$ Connectivity

2-Sat

Card Shuffling

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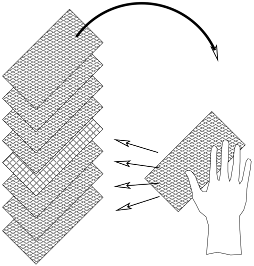

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A *Strong Stationary Time* for a Markov Chain (X_t) with stationary distribution π is a stopping time τ, possibly depending on the stating state x, such that

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for any $A \subset \mathcal{I}$. We can take $\sup_{A \subset \mathcal{I}}$ to complete the result. $\square$

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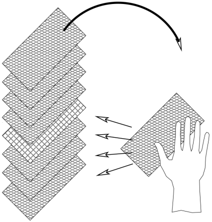

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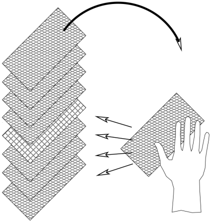

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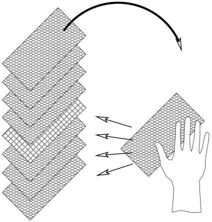

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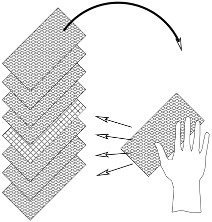

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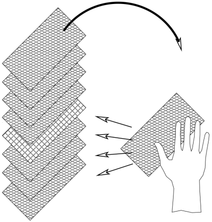

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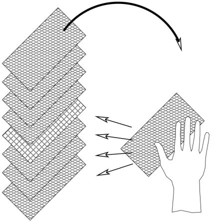

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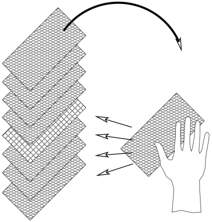

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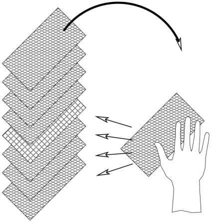

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Thus at time $\tau_{top} - 1$ B sits on the top of a uniform permutation of $[n] \setminus \{B\}$, then we place B in at random so $\mathbf{P}\left[X_{\tau_{top}} \mid \tau_{top} = t \right] = 1/n!$. □

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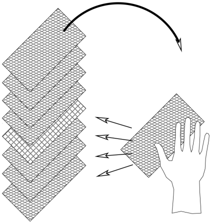

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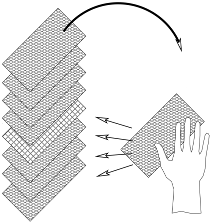

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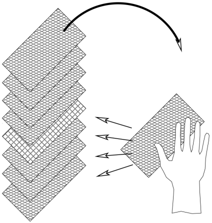

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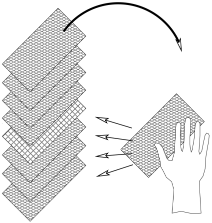

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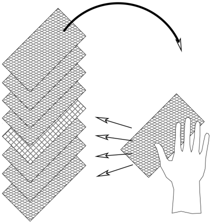

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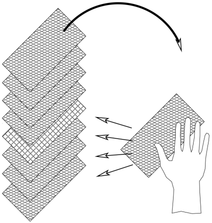
$$t_{mix} \approx \ln\left(|\Sigma_n|\right).$$



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t	≤ 4	5	6	7	8	9
$\Delta(t)$	1.00	.92	.61	.33	.17.	.09

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$s - t$ Connectivity

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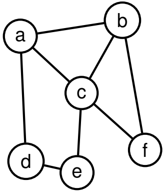

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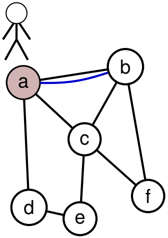

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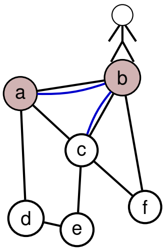

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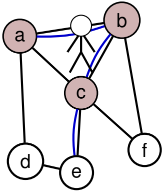

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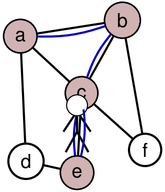

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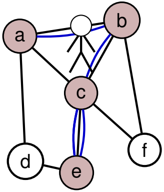

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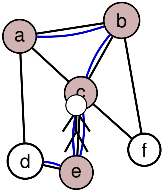

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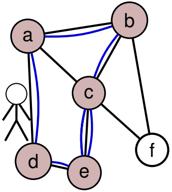

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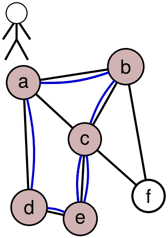

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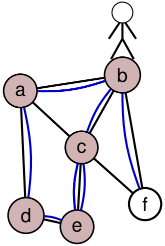

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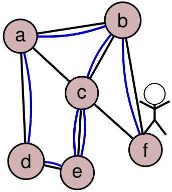

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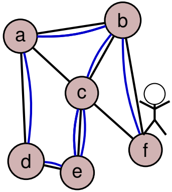

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Matthews bound

For any graph G we have

$$t_{cov}(G) \leq \left(\sum_{m=1}^{n-1} \frac{1}{m} \right) \cdot \max_{x,y \in V} h_{x,y} \approx (\ln n) \cdot \max_{x,y \in V} h_{x,y}.$$

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The n-path P_n is the graph with $V(P_n) = [n]$ and $E(P_n) = \{ij : j = i + 1\}$.

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For the SRW on P_n we have $h_{k,n} = n^2 - k^2$, for any $0 \leq k \leq n$.

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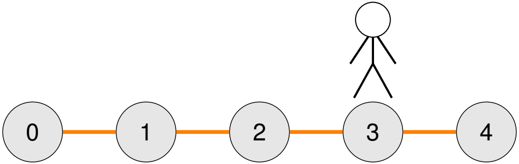

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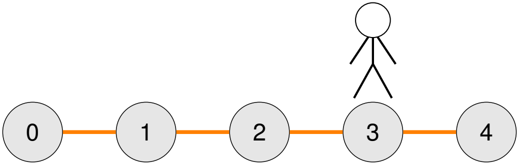

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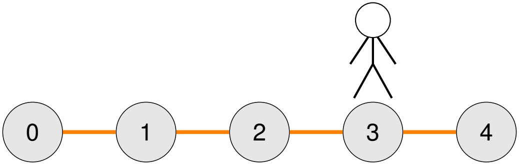

Outline

Shuffling and Strong Stationary Times

Covertime

$s - t$ Connectivity

2-Sat

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Example:

SAT: $(x_1 \vee \overline{x_2} \vee \overline{x_3}) \wedge (\overline{x_1} \vee \overline{x_3}) \wedge (x_1 \vee x_2 \vee x_4) \wedge (x_4 \vee \overline{x_3}) \wedge (x_4 \vee \overline{x_1})$

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RAND 2-SAT Algorithm

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F T T T F F F T F T

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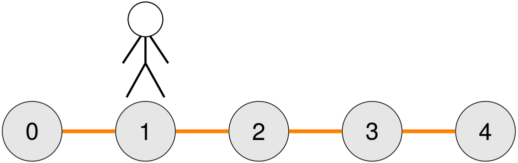

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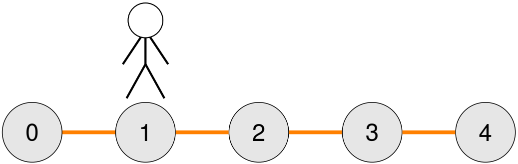

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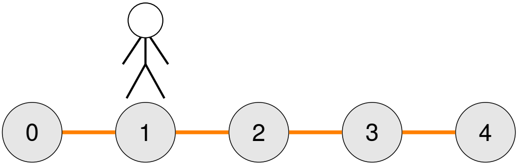

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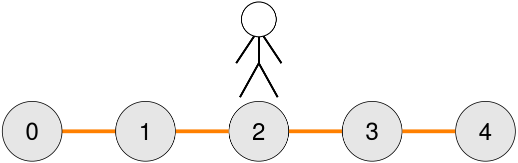

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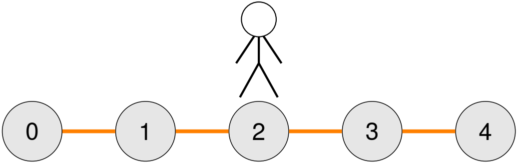

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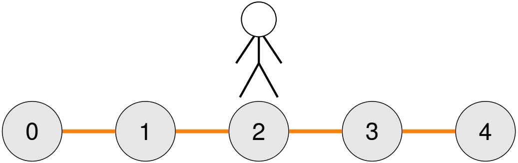

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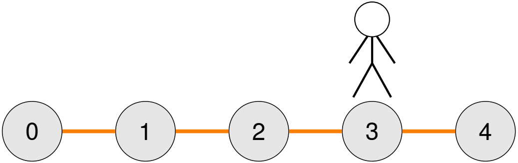

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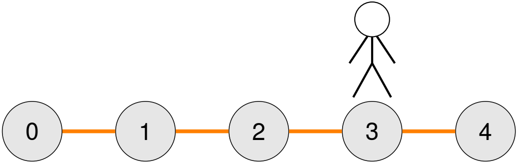

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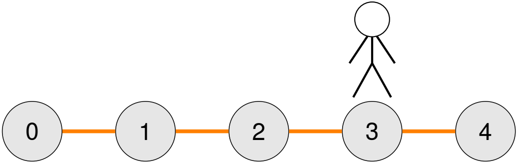

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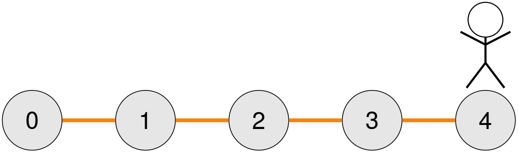

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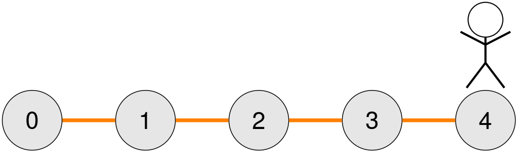

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(3) If a valid solution is found `return` it. O/W `return` unsatisfiable

- Call each loop of (2) a *Step*. Let A_i be the variable assignment at step i.
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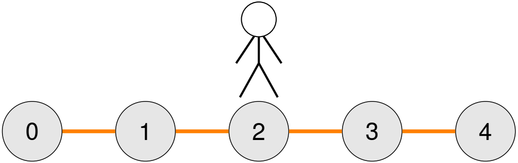
$$(x_1 \vee \overline{x_2}) \wedge (\overline{x_1} \vee \overline{x_3}) \wedge (x_1 \vee x_2) \wedge (x_4 \vee x_3) \wedge (x_4 \vee \overline{x_1})$$

F T T T F F F F F T

$$S = (\mathtt{T}, \mathtt{F}, \mathtt{F}, \mathtt{T}).$$



t	x_1	x_2	x_3	x_4
0	F	F	F	F

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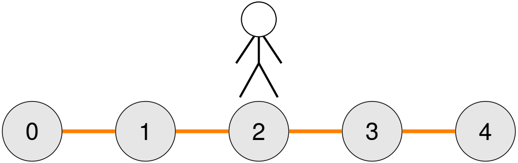

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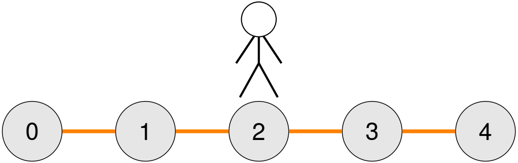

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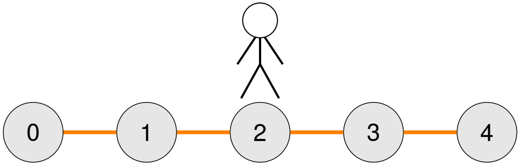

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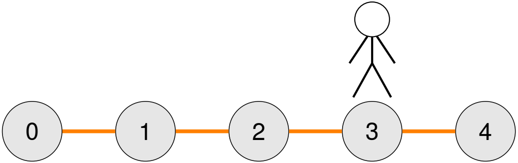

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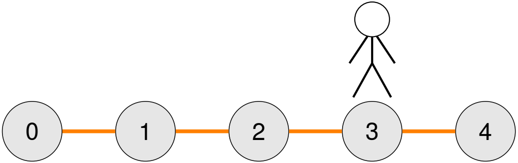

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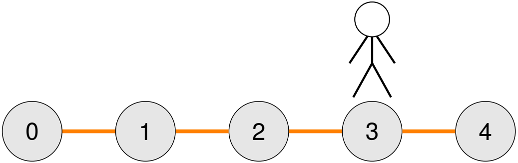

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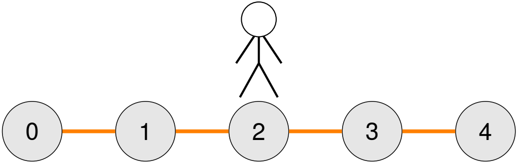

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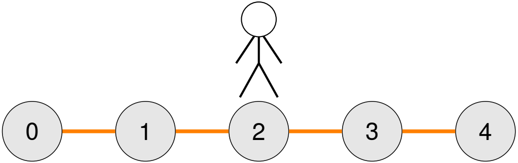

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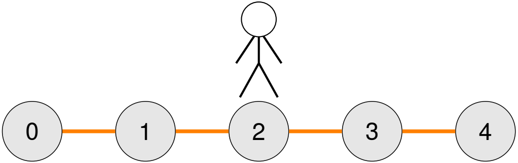

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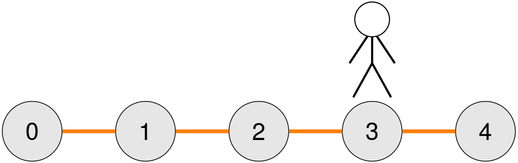

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Example 2 : Solution Found

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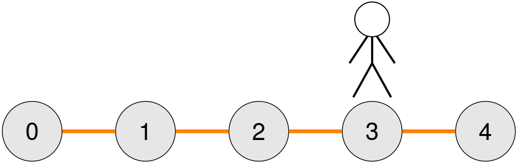

2-SAT and the SRW on the path

Expected iterations of (2) in RAND 2-SAT

If a valid solution S exists then the expected number of iterations of loop (2) before RAND 2-SAT outputs a valid solution is at most n^2.

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Proposition

Provided a solution exists the RAND 2-SAT Algorithm will return a valid solution in time $2n^2$ with probability at least $1/2$.