Lecture 13-14: Sublinear-Time Algorithms

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UNIVERSITY OF CAMBRIDGE

Outline

Sublinear Algorithms Overview

Sublinear Algorithms: Algorithms that return reasonably good approximate answers without scanning or storing the entire input

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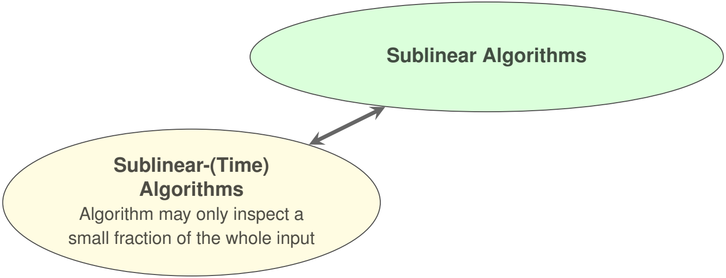

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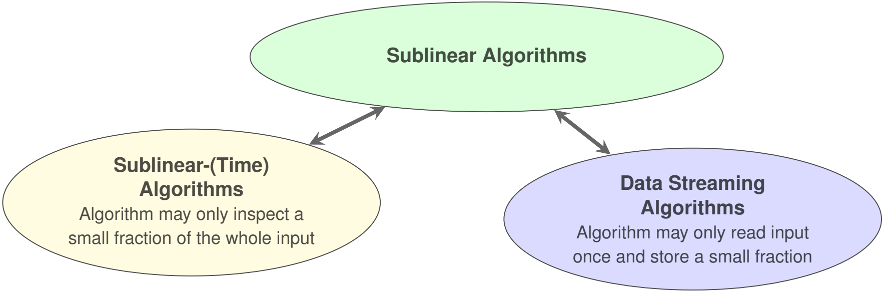

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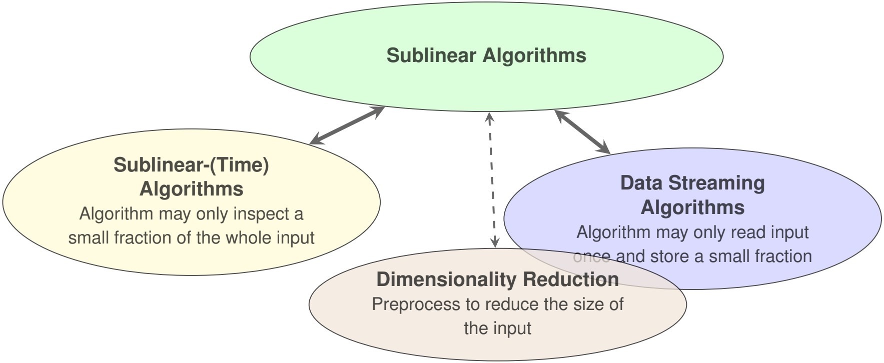

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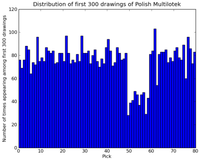

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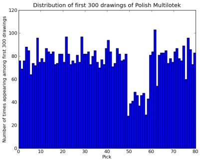

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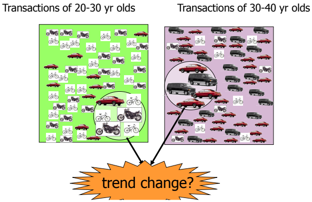

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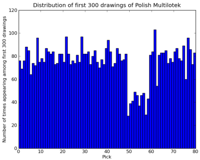

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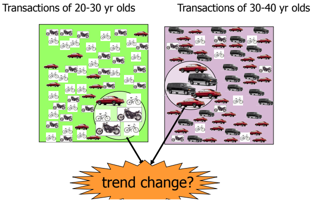

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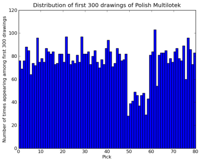
- Health (are there correlations between zip code and health condition?)



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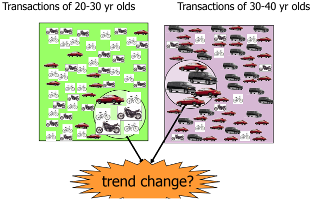

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Testing Probability Distribution (Formal Model)

Model

- Given one (or more) probability distribution $p = (p_1, p_2, \ldots, p_n)$
- distribution(s) are unknown, but can obtain independent samples
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Questions:

1. Is the distribution p close to the uniform distribution u?
2. Is the distribution p close to some other distribution q?
3. What is $\max_{1 \leq i \leq n} p_i$ (heavy hitter)?
4. Are the distributions p and q independent? ...

Testing Uniformity

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Distance between Discrete Distributions

Let p and q be any two distributions over $\{1, 2, \ldots, n\}$. Then:

1. L_1-distance: $\|p - q\|_1 = \sum_{i=1}^{n} |p_i - q_i| \in [0, 2]$,
2. L_2-distance: $\|p - q\|_2 = \sqrt{\sum_{i=1}^{n} (p_i - q_i)^2} \in [0, \sqrt{2}]$,
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3. $p = (\underbrace{2/n, \ldots, 2/n}_{n/2 \text{ times}}, 0, \ldots, 0)$ and $q = (0, \ldots, 0, \underbrace{2/n, \ldots, 2/n}_{n/2 \text{ times}})$. Then $\|p - q\|_1 = 2$, $\|p - q\|_2 = \sqrt{2 \cdot (n/2) \cdot (2/n)^2} = \sqrt{4/n}$ and $\|p - q\|_\infty = 2/n$.

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Disjoint distributions, yet L_2 and L_∞ distances are small!

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Testing Uniformity in the L_1-distance

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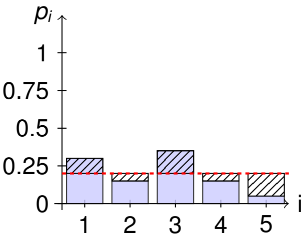

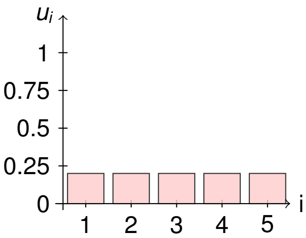

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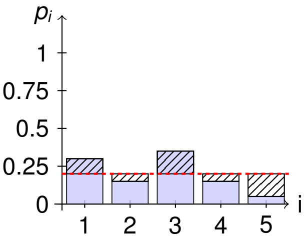
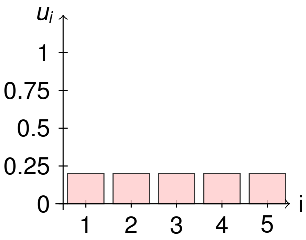
- tester efficient (sub-linear) $\rightsquigarrow$ different from standard statistical tests!
- tester is allowed to have two-sided error
- there is a "grey area" when p is different from but close to uniform, where the tester may give any result





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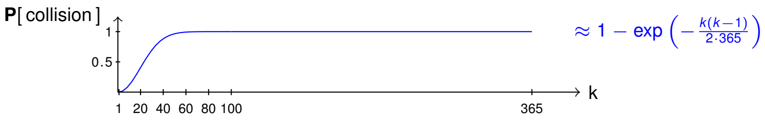
- If p is (close to) uniform, expect to see collisions after $\approx \sqrt{n}$ samples
- If p is far from uniform, expect to see collisions with ??



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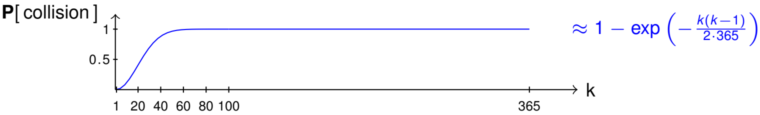

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2. For each $1 \leq i < j \leq r$,

$$\sigma_{i,j} := \begin{cases} 1 & \text{if } x_i = x_j, \\ 0 & \text{otherwise.} \end{cases}$$

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    $$\sum_{1 \leq i < j \leq r} \sigma_{i,j} \stackrel{(\star)}{=} \sum_{k=1}^{n} \binom{a_k}{2}$$

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  - Evaluating $\sum_{1 \leq i < j \leq r} \sigma_{i,j}$ directly takes time quadratic in r
  - Linear-Time Solution:

    1. Maintain array $A = (a_1, a_2, \ldots, a_n)$, where $a_i \in [0, r]$ counts the frequency of samples of item i
    2. Use formula

    $$\sum_{1 \leq i < j \leq r} \sigma_{i,j} \stackrel{(\star)}{=} \sum_{k=1}^{n} \binom{a_k}{2}$$

    3. Since at most $O(r)$ elements in A will be non-zero, using hash-function allows computation in time $O(r)$

Proof of $(\star)$:

$$\sum_{1 \leq i < j \leq r} \sigma_{i,j} = \sum_{1 \leq i < j \leq r} \mathbf{1}_{x_i = x_j}$$

$$= \sum_{1 \leq i < j \leq r} \sum_{k=1}^{n} \mathbf{1}_{x_i = x_j = k} = \sum_{k=1}^{n} \sum_{1 \leq i < j \leq r} \mathbf{1}_{x_i = x_j = k} = \sum_{k=1}^{n} \binom{a_k}{2}. \quad \square$$

Approximation Analysis

Analysis

For any value $r \geq 42 \cdot \frac{\sqrt{n}}{\epsilon^2}$, the algorithm returns a value Y such that

$$\mathbf{P}\left[\,\left|Y - \|p\|_2^2\right| \geq \epsilon \cdot \|p\|_2^2\,\right] \leq 1/3.$$

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Proof (1/5):

- Let us start by computing $\mathbf{E}[\,Y\,]$:

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- Analysis of the deviation more complex (see next slides):

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  - requires a careful analysis of the variance
    (note that the $\sigma_{i,j}$'s are not even pairwise independent! - **Exercise**)

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- Analysis of the deviation more complex (see next slides):
  - requires a careful analysis of the variance (note that the $\sigma_{i,j}$'s are not even pairwise independent! - **Exercise**)
  - final step is an application of Chebysheff's inequality

Approximation Analysis

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Proof (2/5):

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- Define $\widehat{\sigma}_{i,j} := \sigma_{i,j} - \mathbf{E}[\,\sigma_{i,j}\,]$. Note $\mathbf{E}[\,\widehat{\sigma}_{i,j}\,] = 0$, $\widehat{\sigma}_{i,j} \leq \sigma_{i,j}$ and

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$$\mathbf{Var}\left[\sum_{1 \leq i < j \leq r} \sigma_{i,j} \right] = \mathbf{E}\left[\left(\sum_{1 \leq i < j \leq r} \sigma_{i,j} - \sum_{1 \leq i < j \leq r} \mathbf{E}[\sigma_{i,j}] \right)^2 \right] = \mathbf{E}\left[\left(\sum_{1 \leq i < j \leq r} \widehat{\sigma}_{i,j} \right)^2 \right].$$

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- Expanding yields:

There are $2 \cdot \binom{3}{2} = 6$ combinations!

$$\underbrace{\sum_{1\leq i<j\leq r} \mathbf{E}\left[\widehat{\sigma}_{i,j}^2\right]}_{=A} + \underbrace{\sum_{i,j,k,\ell \text{ diff.}} \mathbf{E}[\widehat{\sigma}_{i,j} \cdot \widehat{\sigma}_{k,\ell}]}_{=B} + 6 \cdot \underbrace{\sum_{1\leq i<j<k\leq r} \mathbf{E}[\widehat{\sigma}_{i,j} \cdot \widehat{\sigma}_{j,k}]}_{=C}.$$

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Proof (4/5):

- We have just shown that:

$$\mathbf{Var}\left[\sum_{1 \leq i < j \leq r} \sigma_{i,j}\right] = A + B + 6C$$

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&= \binom{r}{2} \cdot \|p\|_2^2 + 0 + 6 \cdot \frac{\sqrt{3}}{2} \left(\binom{r}{2} \|p\|_2^2 \right)^{3/2} \\
&\leq 5 \left(\binom{r}{2} \|p\|_2^2 \right)^{3/2}
\end{aligned}$$

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For any value $r \geq 42 \cdot \frac{\sqrt{n}}{\epsilon^2}$, the algorithm returns a value Y such that

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Approximation of $\|p - u\|_1$ using $\|p\|_2^2$

UNIFORM-TEST

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Exercise: Prove that any testing algorithm in this model will have a two-sided error!

Analysis of UNIFORM-TEST (1/2)

Case 1: p is uniform.

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Outline

Introduction

Upper Bounds on Testing Uniformity

Lower Bounds on Testing Uniformity

Extensions

Lower Bound

Theorem

Let $0 < \epsilon < 1$. There is no algorithm with the following three properties:

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Idea is that algorithm needs enough samples of the first $\epsilon \cdot n$ elements to see any collisions!

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Exercise: Can you see why is it important to choose $\mathcal{I}$ **randomly**?

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Exercise (in Class)

What is the **expected number of collisions** among $\mathcal{I}$, if we are sampling t times from the distribution p as described on the previous slide?

Let Z denote the number of collisions, so $Z = \sum_{1 \leq k < \ell \leq t} \mathbf{1}_{x_k = x_\ell \wedge x_k \in \mathcal{I}}$. Taking expectations and using linearity of expectation yields:

$$
\begin{aligned}
\mathbf{E}[\,Z\,] = \mathbf{E}\left[\sum_{1 \leq k < \ell \leq t} \mathbf{1}_{x_k = x_\ell \wedge x_k \in \mathcal{I}}\right] &= \sum_{1 \leq k < \ell \leq t} \mathbf{E}\left[\,\mathbf{1}_{x_k = x_\ell \wedge x_k \in \mathcal{I}}\,\right] \\
&= \sum_{1 \leq k < \ell \leq t} \mathbf{P}[\,x_k = x_\ell \wedge x_k \in \mathcal{I}\,] \cdot 1 \\
&= \sum_{1 \leq k < \ell \leq t} \sum_{i=1}^{\epsilon n} \mathbf{P}[\,x_k = x_\ell \wedge x_k \in \mathcal{I} \wedge x_k = i\,] \\
&= \sum_{1 \leq k < \ell \leq t} \sum_{i=1}^{\epsilon n} \mathbf{P}[\,i \in \mathcal{I}\,] \cdot \mathbf{P}[\,x_k = x_\ell = i \mid i \in \mathcal{I}\,] \\
&= \sum_{1 \leq k < \ell \leq t} \sum_{i=1}^{\epsilon n} \frac{1}{2} \cdot \left(\frac{2}{n}\right)^2 = \binom{t}{2} \cdot \epsilon n \cdot \left(\frac{2}{n}\right)^2.
\end{aligned}
$$

Hence if $t = o(\sqrt{n/\epsilon})$, then $\mathbf{E}[\,Z\,] \to 0$ and thus $\mathbf{P}[\,Z = 0\,] \to 1$, as $n \to \infty$.

Outline

Extension 1: Testing Closeness of Arbitrary Distributions (1/2)

$\|p - q\|_2^2$

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APPROXIMATE $\langle p,q\rangle$

1. Sample r elements from p, $x_1, x_2, \ldots, x_r \in [n]$, and sample r elements from q, $y_1, y_2, \ldots, y_r \in [n]$
2. For each $1 \leq i < j \leq r$,

$$\tau_{i,j} := \begin{cases} 1 & \text{if } x_i = y_j, \\ 0 & \text{otherwise.} \end{cases}$$

3. Output $Y := \frac{1}{r^2}\sum_{1\leq i,j\leq r}\tau_{i,j}$.

Extension 1: Testing Closeness of Arbitrary Distributions (2/2)

Theorem (Batu, Fortnow, Rubinfeld, Smith, White; JACM 60(1), 2013)

There exists an algorithm using $\mathcal{O}(1/\epsilon^4)$ samples such that if the distributions p and q satisfy $\|p - q\|_2 \leq \epsilon/2$, then the algorithm accepts with probability at least $2/3$. If $\|p - q\|_2 \geq \epsilon$, then the algorithm rejects with probability at least $2/3$.

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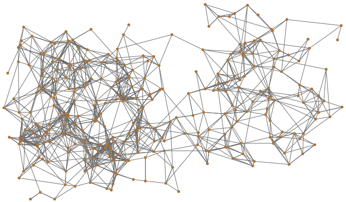
	L_2-distance	L_1-distance
Testing uniformity $\|p-u\|$	$\Theta(1)$	$\Theta(\sqrt{n})$
Testing closeness $\|p-q\|$	$\Theta(1)$	$\in [\Omega(n^{2/3}), \mathcal{O}(n^{2/3} \log n)]$

Figure: Overview of the known sampling complexities for constant $\epsilon \in (0,1)$.

Extension 2: Testing Conductance of Graphs

Testing Conductance of Graphs

- Idea: Start several random walks from the same vertex



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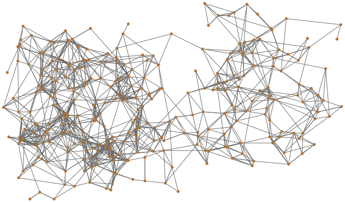

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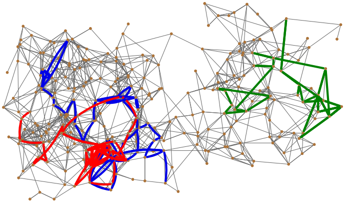
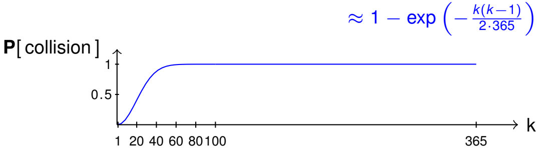
Testing Conductance of Graphs

- Idea: Start several random walks from the same vertex
- Count the number of pairwise collisions among the endpoints of the walks
  - If the number of collisions high, graphs is not an expander
  - If the number of collisions is sufficiently small, graph is an expander



Application 3: Estimating Population Sizes using Mark & Recapture



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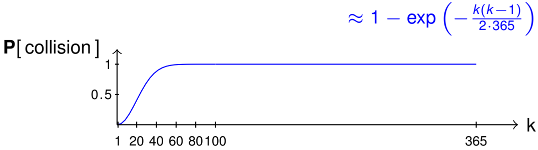

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Mark & Recapture Method:

- First phase: A portion of the population is captured, marked and released
- Second phase: Another portion is captured and the number of marked individuals is counted

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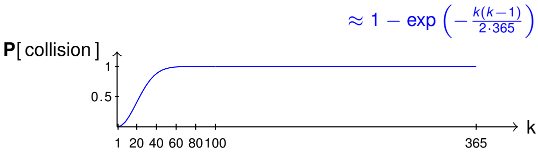

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Mark & Recapture Method:

- First phase: A portion of the population is captured, marked and released
- Second phase: Another portion is captured and the number of marked individuals is counted

  - Essentially the same as collision sampling
  - Can be used to estimate the size of a large network if each node has a unique ID (within an unknown range)