Probability and Computation: Homework Assessment

Submit by 2pm Thursday 24th Jan via moodle or at student reception

Question 1. *Let A, B be independent uniformly random subsets of $[n] := \{1, \dots, n\}$.*

(i) Find $\mathbf{P}[\, A \subseteq B \,]$.

(ii) What is the distribution of $|A|$?

(iii) How about the distribution of $|A \backslash B|$?

(iv) How can you solve part (i) using part (iii)?

Question 2. *Let $X \geq 0$ be an integer valued random variable.*

(i) Show that $\mathbf{E}[\, X \,] = \sum_{i=0}^{\infty} \mathbf{P}[\, X > i \,]$.

(ii) Find a similar expression for $\mathbf{E}\big[\, X^2 \,\big]$.

Question 3. *Throw two fair dice and consider the following three events:*

$$A := \{\text{the sum of the dice is } 7\}, \quad B := \{\text{the first dice rolled a } 3\}, \quad C := \{\text{the second dice rolled a } 4\}.$$

(i) Show that the events are pairwise independent.

(ii) Are the three events are independent?

Question 4. *Suppose you are throwing an unbiased, 6-faced dice sequentially until a 6 turns up followed by a 5.*

(i) What is the expected waiting time?

(ii) What happens if you are waiting for a 6 followed by a 6?

(iii) Explain the difference.

Question 5. *For an event $\mathcal{E}_n$ say that $\mathcal{E}_n$ occurs with high probability (w.h.p.) if $\mathbf{P}[\, \mathcal{E}_n \,] = 1 - o(1)$. Let $\{X_n\}_{n \geq 0}$ be a sequence of non-negative random variables. Show that if $\lim_{n \to \infty} \mathbf{E}[\, X_n \,] = 0$ then $X_n = 0$ w.h.p..*

Question 6. *Let X be a random variable with expected value $\mu < \infty$ and variance $0 < \sigma^2 < \infty$.*

(i) Show that for any real number $k > 0$ we have $\mathbf{P}[\, |X - \mu| \geq k\sigma \,] \leq 1/k^2$.

(ii) Deduce that if $X \geq 0$ then $\mathbf{P}[\, X = 0 \,] \leq \sigma^2/\mu^2$.

(iii) Let $X = X_1 + \cdots + X_n$ where X_i are indicator random variables with $\mathbf{P}[\, X_i = 1 \,] = p_i$ and $\mathbf{P}[\, X_i = 0 \,] = 1 - p_i$. Show that

$$\mathbf{Var}\,[\, X \,] \leq \mathbf{E}[\, X \,] + \sum_{1 \leq i \neq j \leq n} \mathbf{Cov}\,[\, X_i, X_j \,], \quad \textit{where} \quad \mathbf{Cov}\,[\, X_i, X_j \,] = \mathbf{E}[\, X_i X_j \,] - \mathbf{E}[\, X_i \,]\, \mathbf{E}[\, X_j \,].$$

Question 7. *Let $G = (V, E)$ be an undirected graph of $|V| = n$ vertices. Assume that G has been constructed at random according to the following procedure: for any pair of vertices $\{u, v\} \in V \times V$, we put an undirected edge between u and v with probability p (with probability $1 - p$ there is no edge between u and v). We call a vertex u isolated if there is no edge incident to u.*

(i) Let X be the number of isolated vertices in G. Find an expression (which might depend on n and p) for $\mathbf{E}[\, X \,]$.

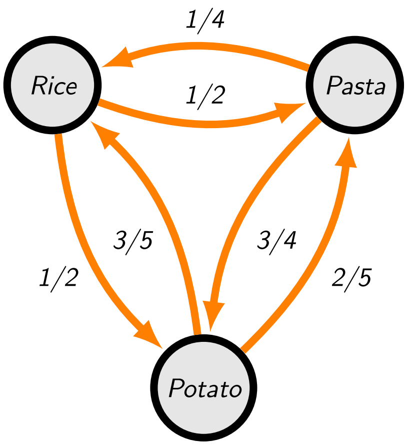
(ii) *Use Question 5 to show that if* $p > \frac{\ln n}{n}$ *then with high probability there are no isolated vertices in* G*. You might need to use the fact that* $\lim_{n\to\infty}\left(1 - \frac{x}{n}\right)^n = \mathrm{e}^{-x}$.

(iii) *Use Question 6 (ii) and (iii) to argue that* $\lim_{n\to\infty} \mathbf{P}[\,X > 0\,] = 1$ *whenever* $p < \frac{\ln n}{n}$*. That is,* G *contains at least one isolated vertex with high probability.*

Question 8. *Let* $\mathcal{F}$ *be a finite collection of binary strings of finite lengths and assume no member of* $\mathcal{F}$ *is a prefix of another one. Let* N_i *denote the number of strings of length* i *in* F*. Prove that*

$$\sum_i \frac{N_i}{2^i} \leq 1.$$

Question 9. *The "College Carbs" Markov chain below makes an appearance in Lecture 2:*



This has transition matrix:

$$P = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/4 & 0 & 3/4 \\ 3/5 & 2/5 & 0 \end{pmatrix}$$

(i) *If I had Pasta on Monday what is the probability that I have Pasta on Thursday* **and** *Potato on Friday?*

(ii) *If I have Pasta today then how many days should I expect to wait until I have Rice?*

(iii) *If I have Pasta today then how many days should I expect to wait until I next have Pasta again?*

Question 10. *The Gamblers ruin chain appears in Lecture 2. This is a Markov Chain on* $\{0, \ldots, n\}$ *with transition matrix* P *is given by* $P_{i,i+1} = a$*,* $P_{i,i-1} = b = 1 - a$ *for* $1 \leq i \leq n-1$ *and* $P_{0,0} = 1$*,* $P_{n,n} = 1$.

(i) *Describe the vectors* v *such that* $vP = v$.

(ii) *Why does this not contradict the results on the stationary distribution given in Lecture 2?*