Logic and Proof

Computer Science Tripos Part IB
Lent Term

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Introduction to Logic

Logic concerns statements in some language.

The language can be natural (English, Latin, ...) or formal.

Some statements are true, others false or meaningless.

Logic concerns relationships between statements: consistency, entailment, ...

Logical proofs model human reasoning (supposedly).

Statements

Statements are declarative assertions:

Black is the colour of my true love's hair.

They are not greetings, questions or commands:

What is the colour of my true love's hair?

I wish my true love had hair.

Get a haircut!

Schematic Statements

Now let the variables X, Y, Z, . . . range over ‘real’ objects

> Black is the colour of X’s hair.
>
> Black is the colour of Y.
>
> Z is the colour of Y.

Schematic statements can even express questions:

> What things are black?

Interpretations and Validity

An interpretation maps variables to real objects:

The interpretation $Y \mapsto$ coal satisfies the statement

> Black is the colour of Y.

but the interpretation $Y \mapsto$ strawberries does not!

A statement A is valid if all interpretations satisfy A.

Consistency, or Satisfiability

A set S of statements is consistent if some interpretation satisfies all elements of S at the same time. Otherwise S is inconsistent.

Examples of inconsistent sets:

$$\{X \text{ part of } Y,\ Y \text{ part of } Z,\ X \text{ NOT part of } Z\}$$

$$\{n \text{ is a positive integer},\ n \neq 1,\ n \neq 2,\ \ldots\}$$

Satisfiable means the same as consistent.

Unsatisfiable means the same as inconsistent.

Entailment, or Logical Consequence

A set S of statements **entails** A if every interpretation that satisfies all elements of S, also satisfies A. We write $S \models A$.

$$\{X \text{ part of } Y,\ \ Y \text{ part of } Z\} \models X \text{ part of } Z$$

$$\{n \neq 1,\ n \neq 2,\ \ldots\} \models n \text{ is NOT a positive integer}$$

$S \models A$ if and only if $\{\neg A\} \cup S$ is inconsistent.

If S is inconsistent, then $S \models A$ for any A.

$\models A$ if and only if A is valid, if and only if $\{\neg A\}$ is inconsistent.

Inference: Proving a Statement

We want to show that A is valid. We can't test infinitely many cases.

Let $\{A_1, \ldots, A_n\} \models B$. If $A_1, \ldots, A_n$ are true then B must be true.

Write this as the inference rule

$$\frac{A_1 \qquad \ldots \qquad A_n}{B}$$

We can use inference rules to construct finite proofs!

Schematic Inference Rules

$$\frac{X \text{ part of } Y \qquad Y \text{ part of } Z}{X \text{ part of } Z}$$

- A proof is correct if it has the right syntactic form, regardless of
- Whether the conclusion is desirable
- Whether the premises or conclusion are true
- Who (or what) created the proof

Why Should we use a Formal Language?

Consider this 'definition': (Berry's paradox)

> The smallest positive integer not definable using nine words

Greater than The number of atoms in the Milky Way galaxy

This number is so large, it is greater than itself!

- A formal language prevents ambiguity.

Survey of Formal Logics

propositional logic is traditional boolean algebra.

first-order logic can say for all and there exists.

higher-order logic reasons about sets and functions.

modal/temporal logics reason about what must, or may, happen.

type theories support constructive mathematics.

All have been used to prove correctness of computer systems.

Syntax of Propositional Logic

$P, Q, R, \ldots$	propositional letter
t	true
f	false
$\neg A$	not A
$A \wedge B$	A and B
$A \vee B$	A or B
$A \rightarrow B$	if A then B
$A \leftrightarrow B$	A if and only if B

Semantics of Propositional Logic

$\neg$, $\wedge$, $\vee$, $\rightarrow$ and $\leftrightarrow$ are truth-functional: functions of their operands.

A	B	$\neg A$	$A \wedge B$	$A \vee B$	$A \rightarrow B$	$A \leftrightarrow B$
1	1	0	1	1	1	1
1	0	0	0	1	0	0
0	1	1	0	1	1	0
0	0	1	0	0	1	1

Interpretations of Propositional Logic

An interpretation is a function from the propositional letters to $\{1, 0\}$.

Interpretation I satisfies a formula A if it evaluates to 1 (true).

Write $\models_I A$

A is valid (a tautology) if every interpretation satisfies A.

Write $\models A$

S is satisfiable if some interpretation satisfies every formula in S.

Implication, Entailment, Equivalence

$A \to B$ means simply $\neg A \vee B$.

$A \models B$ means if $\models_I A$ then $\models_I B$ for every interpretation I.

$A \models B$ if and only if $\models A \to B$.

Equivalence

$A \simeq B$ means $A \models B$ and $B \models A$.

$A \simeq B$ if and only if $\models A \leftrightarrow B$.

Equivalences

$$A \land A \simeq A$$

$$A \land B \simeq B \land A$$

$$(A \land B) \land C \simeq A \land (B \land C)$$

$$A \lor (B \land C) \simeq (A \lor B) \land (A \lor C)$$

$$A \land \mathbf{f} \simeq \mathbf{f}$$

$$A \land \mathbf{t} \simeq A$$

$$A \land \neg A \simeq \mathbf{f}$$

Dual versions: exchange $\land$ with $\lor$ and $\mathbf{t}$ with $\mathbf{f}$ in any equivalence

Negation Normal Form

1. Get rid of $\leftrightarrow$ and $\rightarrow$, leaving just $\wedge$, $\vee$, $\neg$:

$$A \leftrightarrow B \simeq (A \rightarrow B) \wedge (B \rightarrow A)$$

$$A \rightarrow B \simeq \neg A \vee B$$

2. Push negations in, using de Morgan's laws:

$$\neg\neg A \simeq A$$

$$\neg(A \wedge B) \simeq \neg A \vee \neg B$$

$$\neg(A \vee B) \simeq \neg A \wedge \neg B$$

From NNF to Conjunctive Normal Form

3. Push disjunctions in, using distributive laws:

$$A \lor (B \land C) \simeq (A \lor B) \land (A \lor C)$$

$$(B \land C) \lor A \simeq (B \lor A) \land (C \lor A)$$

4. Simplify:

- Delete any disjunction containing P and $\neg P$
- Delete any disjunction that includes another: for example, in $(P \lor Q) \land P$, delete $P \lor Q$.
- Replace $(P \lor A) \land (\neg P \lor A)$ by A

Converting a Non-Tautology to CNF

$$P \vee Q \to Q \vee R$$

1. Elim $\to$: $\neg(P \vee Q) \vee (Q \vee R)$
2. Push $\neg$ in: $(\neg P \wedge \neg Q) \vee (Q \vee R)$
3. Push $\vee$ in: $(\neg P \vee Q \vee R) \wedge (\neg Q \vee Q \vee R)$
4. Simplify: $\neg P \vee Q \vee R$

Not a tautology: try $P \mapsto$ **t**, $Q \mapsto$ **f**, $R \mapsto$ **f**

Tautology checking using CNF

$$((P \to Q) \to P) \to P$$

1. Elim $\to$: $\neg[\neg(\neg P \lor Q) \lor P] \lor P$
2. Push $\neg$ in: $[\neg\neg(\neg P \lor Q) \land \neg P] \lor P$

   $[(\neg P \lor Q) \land \neg P] \lor P$
3. Push $\lor$ in: $(\neg P \lor Q \lor P) \land (\neg P \lor P)$
4. Simplify: $\mathbf{t} \land \mathbf{t}$

   $\mathbf{t}$ *It's a tautology!*

A Simple Proof System

Axiom Schemes

K $A \to (B \to A)$

S $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$

DN $\neg\neg A \to A$

Inference Rule: Modus Ponens

$$\frac{A \to B \qquad A}{B}$$

A Simple (?) Proof of $A \to A$

$$(A \to ((D \to A) \to A)) \to \qquad (1)$$
$$((A \to (D \to A)) \to (A \to A)) \quad \text{by S}$$

$$A \to ((D \to A) \to A) \quad \text{by K} \qquad (2)$$

$$(A \to (D \to A)) \to (A \to A) \quad \text{by MP, (1), (2)} \qquad (3)$$

$$A \to (D \to A) \quad \text{by K} \qquad (4)$$

$$A \to A \quad \text{by MP, (3), (4)} \qquad (5)$$

Some Facts about Deducibility

A is deducible from the set S if there is a finite proof of A starting from elements of S. Write $S \vdash A$.

Soundness Theorem. If $S \vdash A$ then $S \models A$.

Completeness Theorem. If $S \models A$ then $S \vdash A$.

Deduction Theorem. If $S \cup \{A\} \vdash B$ then $S \vdash A \to B$.

Gentzen's Natural Deduction Systems

The context of assumptions may vary.

Each logical connective is defined independently.

The introduction rule for $\land$ shows how to deduce $A \land B$:

$$\frac{A \qquad B}{A \land B}$$

The elimination rules for $\land$ shows what to deduce from $A \land B$:

$$\frac{A \land B}{A} \qquad \frac{A \land B}{B}$$

The Sequent Calculus

Sequent $A_1, \ldots, A_m \Rightarrow B_1, \ldots, B_n$ means,

$$\text{if } A_1 \land \ldots \land A_m \text{ then } B_1 \lor \ldots \lor B_n$$

$A_1, \ldots, A_m$ are assumptions; $B_1, \ldots, B_n$ are goals

Γ and Δ are sets in $\Gamma \Rightarrow \Delta$

$A, \Gamma \Rightarrow A, \Delta$ is trivially true (and is called a basic sequent).

Sequent Calculus Rules

$$\frac{\Gamma \Rightarrow \Delta, A \qquad A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \ (\mathrm{cut})$$

$$\frac{\Gamma \Rightarrow \Delta, A}{\neg A, \Gamma \Rightarrow \Delta} \ (\neg l) \qquad \frac{A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg A} \ (\neg r)$$

$$\frac{A, B, \Gamma \Rightarrow \Delta}{A \land B, \Gamma \Rightarrow \Delta} \ (\land l) \qquad \frac{\Gamma \Rightarrow \Delta, A \qquad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \land B} \ (\land r)$$

More Sequent Calculus Rules

$$\frac{A, \Gamma \Rightarrow \Delta \qquad B, \Gamma \Rightarrow \Delta}{A \lor B, \Gamma \Rightarrow \Delta}\;(\lor l) \qquad \frac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A \lor B}\;(\lor r)$$

$$\frac{\Gamma \Rightarrow \Delta, A \qquad B, \Gamma \Rightarrow \Delta}{A \to B, \Gamma \Rightarrow \Delta}\;(\to l) \qquad \frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \to B}\;(\to r)$$

Easy Sequent Calculus Proofs

$$\dfrac{\dfrac{\overline{A, B \Rightarrow A}}{A \wedge B \Rightarrow A}\,(\wedge l)}{\Rightarrow (A \wedge B) \rightarrow A}\,(\rightarrow r)$$

$$\dfrac{\dfrac{\dfrac{\overline{A, B \Rightarrow B, A}}{A \Rightarrow B, B \rightarrow A}\,(\rightarrow r)}{\Rightarrow A \rightarrow B,\ B \rightarrow A}\,(\rightarrow r)}{\Rightarrow (A \rightarrow B) \vee (B \rightarrow A)}\,(\vee r)$$

Part of a Distributive Law

$$
\dfrac{
  \dfrac{
    \dfrac{}{A \Rightarrow A, B}
    \qquad
    \dfrac{\dfrac{}{B, C \Rightarrow A, B}}{B \land C \Rightarrow A, B}\;(\land l)
  }{
    \dfrac{A \lor (B \land C) \Rightarrow A, B}{A \lor (B \land C) \Rightarrow A \lor B}\;(\lor r)
  }\;(\lor l)
  \qquad \text{similar}
}{A \lor (B \land C) \Rightarrow (A \lor B) \land (A \lor C)}\;(\land r)
$$

Second subtree proves $A \lor (B \land C) \Rightarrow A \lor C$ similarly

A Failed Proof

$$\dfrac{\dfrac{A \Rightarrow B, C \qquad \overline{B \Rightarrow B, C}}{A \vee B \Rightarrow B, C}\,(\vee l)}{\dfrac{A \vee B \Rightarrow B \vee C}{\Rightarrow (A \vee B) \to (B \vee C)}\,(\to r)}\,(\vee r)$$

$A \mapsto \mathbf{t}$, $B \mapsto \mathbf{f}$, $C \mapsto \mathbf{f}$ falsifies unproved sequent!

Outline of First-Order Logic

Reasons about functions and relations over a set of individuals:

$$\frac{\mathrm{father}(\mathrm{father}(x)) = \mathrm{father}(\mathrm{father}(y))}{\mathrm{cousin}(x, y)}$$

Reasons about all and some individuals:

$$\frac{\text{All men are mortal} \qquad \text{Socrates is a man}}{\text{Socrates is mortal}}$$

Cannot reason about all functions or all relations, etc.

Function Symbols; Terms

Each function symbol stands for an n-place function.

A constant symbol is a 0-place function symbol.

A variable ranges over all individuals.

A term is a variable, constant or a function application

$$f(t_1, \ldots, t_n)$$

where f is an n-place function symbol and t_1, ..., t_n are terms.

We choose the language, adopting any desired function symbols.

Relation Symbols; Formulae

Each relation symbol stands for an n-place relation.

Equality is the 2-place relation symbol $=$

An atomic formula has the form $R(t_1, \ldots, t_n)$ where R is an n-place relation symbol and t_1, . . ., t_n are terms.

A formula is built up from atomic formulæ using $\neg$, $\wedge$, $\vee$, and so forth.

Later, we can add quantifiers.

The Power of Quantifier-Free FOL

It is surprisingly expressive, if we include strong induction rules.

We can easily prove the equivalence of mathematical functions:

$$
\begin{aligned}
p(z,0) &= 1 \\
p(z,n+1) &= p(z,n) \times z
\end{aligned}
\qquad
\begin{aligned}
q(z,1) &= z \\
q(z,2 \times n) &= q(z \times z, n) \\
q(z,2 \times n + 1) &= q(z \times z, n) \times z
\end{aligned}
$$

The prover ACL2 uses this logic to do major hardware proofs.

Universal and Existential Quantifiers

$\forall x\, A$	for all x, the formula A holds
$\exists x\, A$	there exists x such that A holds

Syntactic variations:

$\forall xyz\, A$	abbreviates $\forall x\, \forall y\, \forall z\, A$
$\forall z \,.\, A \land B$	is an alternative to $\forall z\, (A \land B)$

The variable x is bound in $\forall x\, A$; compare with $\int f(x)\, dx$

The Expressiveness of Quantifiers

All men are mortal:

$$\forall x\, (\mathrm{man}(x) \rightarrow \mathrm{mortal}(x))$$

All mothers are female:

$$\forall x\, \mathrm{female}(\mathrm{mother}(x))$$

There exists a unique x such that A, sometimes written $\exists! x\, A$

$$\exists x\, [A(x) \land \forall y\, (A(y) \rightarrow y = x)]$$

The Point of Semantics

We have to attach meanings to symbols like 1, $+$, $<$, etc.

Why is this necessary? Why can't 1 just mean 1??

The point is that mathematics derives its flexibility from allowing different interpretations of symbols.

- A group has a unit 1, a product $x \cdot y$ and inverse x^{-1}.
- In the most important uses of groups, 1 isn't a number but a 'unit permutation', 'unit rotation', etc.

Constants: Interpreting $\mathrm{mortal}(\mathrm{Socrates})$

An interpretation $\mathcal{I} = (D, I)$ defines the semantics of a first-order language.

D is a non-empty set, called the domain or universe.

I maps symbols to 'real' elements, functions and relations:

c a constant symbol	$I[c] \in D$
f an n-place function symbol	$I[f] \in D^n \to D$
P an n-place relation symbol	$I[P] \in D^n \to \{1, 0\}$

Variables: Interpreting father(y)

A valuation $V : \mathrm{Var} \to D$ supplies the values of free variables.

V and $\mathcal{I}$ together determine the value of any term t, by recursion.

This value is written $\mathcal{I}_V[t]$, and here are the recursion rules:

$$\mathcal{I}_V[x] \stackrel{\text{def}}{=} V(x) \qquad \text{if } x \text{ is a variable}$$

$$\mathcal{I}_V[c] \stackrel{\text{def}}{=} I[c]$$

$$\mathcal{I}_V[f(t_1, \ldots, t_n)] \stackrel{\text{def}}{=} I[f](\mathcal{I}_V[t_1], \ldots, \mathcal{I}_V[t_n])$$

Tarski's Truth-Definition

An interpretation $\mathcal{I}$ and valuation function V similarly specify the truth value (1 or 0) of any formula A.

Quantifiers are the only problem, as they bind variables.

$V\{a/x\}$ is the valuation that maps x to a and is otherwise like V.

With the help of $V\{a/x\}$, we now formally define $\models_{\mathcal{I},V} A$, the truth value of A.

The Meaning of Truth—In FOL!

For interpretation $\mathcal{I}$ and valuation V, define $\models_{\mathcal{I},V}$ by recursion.

$\models_{\mathcal{I},V} P(t)$	if $I[P](\mathcal{I}_V[t])$ equals 1 (is true)
$\models_{\mathcal{I},V} t = u$	if $\mathcal{I}_V[t]$ equals $\mathcal{I}_V[u]$
$\models_{\mathcal{I},V} A \land B$	if $\models_{\mathcal{I},V} A$ and $\models_{\mathcal{I},V} B$
$\models_{\mathcal{I},V} \exists x\, A$	if $\models_{\mathcal{I},V\{m/x\}} A$ holds for some $m \in D$

Finally, we define

$\models_{\mathcal{I}} A$	if $\models_{\mathcal{I},V} A$ holds for all V.

A closed formula A is satisfiable if $\models_{\mathcal{I}} A$ for some $\mathcal{I}$.

Free vs Bound Variables

All occurrences of x in $\forall x\, A$ and $\exists x\, A$ are bound

An occurrence of x is free if it is not bound:

$$\forall y\, \exists z\, R(y, z, f(y, x))$$

In this formula, y and z are bound while x is free.

We may rename bound variables without affecting the meaning:

$$\forall w\, \exists z'\, R(w, z', f(w, x))$$

Substitution for Free Variables

$A[t/x]$ means substitute t for x in A:

$$
\begin{array}{rcll}
(B \land C)[t/x] & \text{is} & B[t/x] \land C[t/x] & \\
(\forall x\, B)[t/x] & \text{is} & \forall x\, B & \\
(\forall y\, B)[t/x] & \text{is} & \forall y\, B[t/x] & (x \neq y) \\
(P(u))[t/x] & \text{is} & P(u[t/x]) &
\end{array}
$$

When substituting $A[t/x]$, no variable of t may be bound in A!

Example: $(\forall y\ (x = y))\,[y/x]$ is not equivalent to $\forall y\ (y = y)$

Some Equivalences for Quantifiers

$$\neg(\forall x\, A) \simeq \exists x\, \neg A$$

$$\forall x\, A \simeq \forall x\, A \land A[t/x]$$

$$(\forall x\, A) \land (\forall x\, B) \simeq \forall x\, (A \land B)$$

But we do not have $(\forall x\, A) \lor (\forall x\, B) \simeq \forall x\, (A \lor B)$.

Dual versions: exchange $\forall$ with $\exists$ and $\land$ with $\lor$

Further Quantifier Equivalences

These hold only if x is not free in B.

$$(\forall x\, A) \land B \simeq \forall x\, (A \land B)$$

$$(\forall x\, A) \lor B \simeq \forall x\, (A \lor B)$$

$$(\forall x\, A) \to B \simeq \exists x\, (A \to B)$$

These let us expand or contract a quantifier's scope.

Reasoning by Equivalences

$$
\begin{aligned}
\exists x\,(x = a \land P(x)) &\simeq \exists x\,(x = a \land P(a)) \\
&\simeq \exists x\,(x = a) \land P(a) \\
&\simeq P(a)
\end{aligned}
$$

$$
\begin{aligned}
&\exists z\,(P(z) \to P(a) \land P(b)) \\
&\qquad \simeq \forall z\, P(z) \to P(a) \land P(b) \\
&\qquad \simeq \forall z\, P(z) \land P(a) \land P(b) \to P(a) \land P(b) \\
&\qquad \simeq \mathbf{t}
\end{aligned}
$$

Sequent Calculus Rules for $\forall$

$$\frac{A[t/x], \Gamma \Rightarrow \Delta}{\forall x\, A, \Gamma \Rightarrow \Delta}\ (\forall l) \qquad \frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, \forall x\, A}\ (\forall r)$$

Rule $(\forall l)$ can create many instances of $\forall x\, A$

Rule $(\forall r)$ holds provided x is not free in the conclusion!

Not allowed to prove

$$\frac{\overline{P(y) \Rightarrow P(y)}}{P(y) \Rightarrow \forall y\, P(y)}\ (\forall r)$$

This is nonsense!

A Simple Example of the $\forall$ Rules

$$\dfrac{\dfrac{\overline{P(f(y)) \Rightarrow P(f(y))}}{\forall x\, P(x) \Rightarrow P(f(y))}\,(\forall l)}{\forall x\, P(x) \Rightarrow \forall y\, P(f(y))}\,(\forall r)$$

A Not-So-Simple Example of the $\forall$ Rules

$$
\dfrac{\dfrac{\dfrac{\dfrac{\overline{P \Rightarrow Q(y), P} \qquad \overline{P, Q(y) \Rightarrow Q(y)}}{P,\ P \to Q(y) \Rightarrow Q(y)}\ (\to l)}{P,\ \forall x\,(P \to Q(x)) \Rightarrow Q(y)}\ (\forall l)}{P,\ \forall x\,(P \to Q(x)) \Rightarrow \forall y\, Q(y)}\ (\forall r)}{\forall x\,(P \to Q(x)) \Rightarrow P \to \forall y\, Q(y)}\ (\to r)
$$

In $(\forall l)$, we must replace x by y.

Sequent Calculus Rules for $\exists$

$$\frac{A, \Gamma \Rightarrow \Delta}{\exists x\, A, \Gamma \Rightarrow \Delta}\ (\exists l) \qquad \frac{\Gamma \Rightarrow \Delta, A[t/x]}{\Gamma \Rightarrow \Delta, \exists x\, A}\ (\exists r)$$

Rule $(\exists l)$ holds provided x is not free in the conclusion!

Rule $(\exists r)$ can create many instances of $\exists x\, A$

For example, to prove this counter-intuitive formula:

$$\exists z\, (P(z) \to P(a) \land P(b))$$

Part of the $\exists$ Distributive Law

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{P(x) \Rightarrow P(x), Q(x)}}{P(x) \Rightarrow P(x) \lor Q(x)}\,(\lor r)}{P(x) \Rightarrow \exists y\,(P(y) \lor Q(y))}\,(\exists r)}{\exists x\,P(x) \Rightarrow \exists y\,(P(y) \lor Q(y))}\,(\exists l)
\qquad
\dfrac{\text{similar}}{\exists x\,Q(x) \Rightarrow \exists y\ \ldots}\,(\exists l)
$$
$$
\overline{\exists x\,P(x) \lor \exists x\,Q(x) \Rightarrow \exists y\,(P(y) \lor Q(y))}\,(\lor l)
$$

Second subtree proves $\exists x\,Q(x) \Rightarrow \exists y\,(P(y) \lor Q(y))$ similarly

In $(\exists r)$, we must replace y by x.

A Failed Proof

$$\dfrac{\dfrac{\dfrac{\dfrac{P(x), Q(y) \Rightarrow P(x) \land Q(x)}{P(x), Q(y) \Rightarrow \exists z\,(P(z) \land Q(z))}\,(\exists r)}{P(x), \exists x\, Q(x) \Rightarrow \exists z\,(P(z) \land Q(z))}\,(\exists l)}{\exists x\, P(x), \exists x\, Q(x) \Rightarrow \exists z\,(P(z) \land Q(z))}\,(\exists l)}{\exists x\, P(x) \land \exists x\, Q(x) \Rightarrow \exists z\,(P(z) \land Q(z))}\,(\land l)$$

We cannot use $(\exists l)$ twice with the same variable

This attempt renames the x in $\exists x\, Q(x)$, to get $\exists y\, Q(y)$

Clause Form

Clause: a disjunction of literals

$$\neg K_1 \vee \cdots \vee \neg K_m \vee L_1 \vee \cdots \vee L_n$$

Set notation: $\{\neg K_1, \ldots, \neg K_m, L_1, \ldots, L_n\}$

Kowalski notation: $K_1, \cdots, K_m \to L_1, \cdots, L_n$

$L_1, \cdots, L_n \leftarrow K_1, \cdots, K_m$

Empty clause: $\{\}$ or $\square$

Empty clause is equivalent to **f**, meaning contradiction!

Outline of Clause Form Methods

To prove A, obtain a contradiction from $\neg A$:

1. Translate $\neg A$ into CNF as $A_1 \wedge \cdots \wedge A_m$
2. This is the set of clauses $A_1, \ldots, A_m$
3. Transform the clause set, preserving consistency

Deducing the empty clause refutes $\neg A$.

An empty clause set (all clauses deleted) means $\neg A$ is satisfiable.

The basis for SAT solvers and resolution provers.

The Davis-Putnam-Logeman-Loveland Method

1. Delete tautological clauses: $\{P, \neg P, \ldots\}$
2. For each unit clause $\{L\}$,
   - delete all clauses containing L
   - delete $\neg L$ from all clauses
3. Delete all clauses containing pure literals
4. Perform a case split on some literal; stop if a model is found

DPLL is a decision procedure: it finds a contradiction or a model.

DPLL on a Non-Tautology

Consider $P \lor Q \to Q \lor R$

Clauses are $\{P, Q\}$ $\{\neg Q\}$ $\{\neg R\}$

$\{P, Q\}$	$\{\neg Q\}$	$\{\neg R\}$	initial clauses
$\{P\}$		$\{\neg R\}$	unit $\neg Q$
		$\{\neg R\}$	unit P (also pure)
			unit $\neg R$ (also pure)

All clauses deleted! Clauses satisfiable by $P \mapsto \mathbf{t}$, $Q \mapsto \mathbf{f}$, $R \mapsto \mathbf{f}$

Example of a Case Split on P

$\{\neg Q, R\}$ $\{\neg R, P\}$ $\{\neg R, Q\}$ $\{\neg P, Q, R\}$ $\{P, Q\}$ $\{\neg P, \neg Q\}$

$\{\neg Q, R\}$	$\{\neg R, Q\}$	$\{Q, R\}$	$\{\neg Q\}$	if P is true
	$\{\neg R\}$	$\{R\}$		unit $\neg Q$
	$\{\}$			unit R
$\{\neg Q, R\}$	$\{\neg R\}$	$\{\neg R, Q\}$	$\{Q\}$	if P is false
$\{\neg Q\}$			$\{Q\}$	unit $\neg R$
			$\{\}$	unit $\neg Q$

Both cases yield contradictions: the clauses are inconsistent!

SAT solvers in the Real World

- Progressed from joke to killer technology in 10 years.
- Princeton's zChaff has solved problems with more than one million variables and 10 million clauses.
- Applications include finding bugs in device drivers (Microsoft's SLAM project).
- SMT solvers (satisfiability modulo theories) extend SAT solving to handle arithmetic, arrays and bit vectors.

The Resolution Rule

From $B \vee A$ and $\neg B \vee C$ infer $A \vee C$

In set notation,

$$\frac{\{B, A_1, \ldots, A_m\} \qquad \{\neg B, C_1, \ldots, C_n\}}{\{A_1, \ldots, A_m, C_1, \ldots, C_n\}}$$

Some special cases: (remember that $\square$ is just $\{\}$)

$$\frac{\{B\} \qquad \{\neg B, C_1, \ldots, C_n\}}{\{C_1, \ldots, C_n\}} \qquad\qquad \frac{\{B\} \qquad \{\neg B\}}{\square}$$

Simple Example: Proving $P \land Q \to Q \land P$

Hint: use $\neg(A \to B) \simeq A \land \neg B$

1. Negate! $\neg[P \land Q \to Q \land P]$
2. Push $\neg$ in: $(P \land Q) \land \neg(Q \land P)$

   $(P \land Q) \land (\neg Q \lor \neg P)$

Clauses: $\{P\}$ $\{Q\}$ $\{\neg Q, \neg P\}$

Resolve $\{P\}$ and $\{\neg Q, \neg P\}$ getting $\{\neg Q\}$.

Resolve $\{Q\}$ and $\{\neg Q\}$ getting $\square$: we have refuted the negation.

Another Example

Refute $\neg[(P \lor Q) \land (P \lor R) \to P \lor (Q \land R)]$

From $(P \lor Q) \land (P \lor R)$, get clauses $\{P, Q\}$ and $\{P, R\}$.

From $\neg\, [P \lor (Q \land R)]$ get clauses $\{\neg P\}$ and $\{\neg Q, \neg R\}$.

Resolve $\{\neg P\}$ and $\{P, Q\}$ getting $\{Q\}$.

Resolve $\{\neg P\}$ and $\{P, R\}$ getting $\{R\}$.

Resolve $\{Q\}$ and $\{\neg Q, \neg R\}$ getting $\{\neg R\}$.

Resolve $\{R\}$ and $\{\neg R\}$ getting $\square$, contradiction.

The Saturation Algorithm

At start, all clauses are passive. None are active.

1. Transfer a clause (current) from passive to active.
2. Form all resolvents between current and an active clause.
3. Use new clauses to simplify both passive and active.
4. Put the new clauses into passive.

Repeat until contradiction found or passive becomes empty.

Heuristics and Hacks for Resolution

Orderings to focus the search on specific literals

Subsumption, or deleting redundant clauses

Indexing: elaborate data structures for speed

Preprocessing: removing tautologies, symmetries . . .

Weighting: giving priority to “good” clauses over those containing unwanted constants

Reducing FOL to Propositional Logic

NNF :	Eliminate all connectives except $\vee$, $\wedge$ and $\neg$
Skolemize:	Remove quantifiers, preserving consistency
Herbrand models:	Reduce the class of interpretations
Herbrand's Thm:	Contradictions have finite, ground proofs
Unification:	Automatically find the right instantiations

Finally, combine unification with resolution

Skolemization, or Getting Rid of $\exists$

Start with a formula in NNF, with quantifiers nested like this:

$$\forall x_1\,(\cdots\forall x_2\,(\cdots\forall x_k\,(\cdots\exists y\,A\cdots)\cdots)\cdots)$$

Choose a fresh k-place function symbol, say f

Delete $\exists y$ and replace y by $f(x_1,x_2,\ldots,x_k)$. We get

$$\forall x_1\,(\cdots\forall x_2\,(\cdots\forall x_k\,(\cdots A[f(x_1,x_2,\ldots,x_k)/y]\cdots)\cdots)\cdots)$$

Repeat until no $\exists$ quantifiers remain

Example of Conversion to Clauses

For proving $\exists x\,[P(x) \to \forall y\,P(y)]$

$\neg\,[\exists x\,[P(x) \to \forall y\,P(y)]]$	negated goal
$\forall x\,[P(x) \land \exists y\,\neg P(y)]$	conversion to NNF
$\forall x\,[P(x) \land \neg P(f(x))]$	Skolem term $f(x)$
$\{P(x)\} \quad \{\neg P(f(x))\}$	Final clauses

Correctness of Skolemization

The formula $\forall x\, \exists y\, A$ is consistent

$\iff$ it holds in some interpretation $\mathcal{I} = (D, I)$

$\iff$ for all $x \in D$ there is some $y \in D$ such that A holds

$\iff$ some function $\hat{f}$ in $D \to D$ yields suitable values of y

$\iff$ $A[f(x)/y]$ holds in some $\mathcal{I}'$ extending $\mathcal{I}$ so that f denotes $\hat{f}$

$\iff$ the formula $\forall x\, A[f(x)/y]$ is consistent.

The Herbrand Universe for a Set of Clauses S

$$H_0 \stackrel{\text{def}}{=} \text{the set of constants in } S \quad \text{(must be non-empty)}$$

$$H_{i+1} \stackrel{\text{def}}{=} H_i \cup \{f(t_1, \ldots, t_n) \mid t_1, \ldots, t_n \in H_i$$
$$\text{and } f \text{ is an } n\text{-place function symbol in } S\}$$

$$H \stackrel{\text{def}}{=} \bigcup_{i \geq 0} H_i \qquad \text{Herbrand Universe}$$

H_i contains just the terms with at most i nested function applications.

H consists of the terms in S that contain no variables (ground terms).

The Herbrand Semantics of Predicates

An Herbrand interpretation defines an n-place predicate P to denote a truth-valued function in $H^n \to \{1, 0\}$, making $P(t_1, \ldots, t_n)$ true ...

- if and only if the formula $P(t_1, \ldots, t_n)$ holds in our desired “real” interpretation $\mathcal{I}$ of the clauses.
- Thus, an Herbrand interpretation can imitate any other interpretation.

The Inspiration for Clause Methods

Herbrand's Theorem: *Let S be a set of clauses.*

S is unsatisfiable $\iff$ there is a finite unsatisfiable set S' of ground instances of clauses of S.

- **Finite**: we can compute it
- **Instance**: result of substituting for variables
- **Ground**: no variables remain—it's propositional!

Example: S could be $\{P(x)\} \quad \{\neg P(f(y))\}$,
and S' could be $\{P(f(a))\} \quad \{\neg P(f(a))\}$.

Unification

Finding a common instance of two terms. Lots of applications:

- Prolog and other logic programming languages
- Theorem proving: resolution and other procedures
- Tools for reasoning with equations or satisfying constraints
- Polymorphic type-checking (ML and other functional languages)

It is an intuitive generalization of pattern-matching.

Four Unification Examples

$f(x,b)$ $f(a,y)$	$f(x,x)$ $f(a,b)$	$f(x,x)$ $f(y,g(y))$	$j(x,x,z)$ $j(w,a,h(w))$
$f(a,b)$	None	None	$j(a,a,h(a))$
$[a/x, b/y]$	Fail	Fail	$[a/w, a/x, h(a)/z]$

The output is a substitution, mapping variables to terms.

Other occurrences of those variables also must be updated.

Unification yields a most general substitution (in a technical sense).

Theorem-Proving Example 1

$$(\exists y\, \forall x\, R(x, y)) \to (\forall x\, \exists y\, R(x, y))$$

After negation, the clauses are $\{R(x, a)\}$ and $\{\neg R(b, y)\}$.

The literals $R(x, a)$ and $R(b, y)$ have unifier $[b/x, a/y]$.

We have the contradiction $R(b, a)$ and $\neg R(b, a)$.

The theorem is proved by contradiction!

Theorem-Proving Example 2

$$(\forall x\, \exists y\, R(x, y)) \to (\exists y\, \forall x\, R(x, y))$$

After negation, the clauses are $\{R(x, f(x))\}$ and $\{\neg R(g(y), y)\}$.

The literals $R(x, f(x))$ and $R(g(y), y)$ are not unifiable.

(They fail the occurs check.)

We can't get a contradiction. Formula is not a theorem!

The Binary Resolution Rule

$$\frac{\{B, A_1, \ldots, A_m\} \qquad \{\neg D, C_1, \ldots, C_n\}}{\{A_1, \ldots, A_m, C_1, \ldots, C_n\}\sigma} \qquad \text{provided } B\sigma = D\sigma$$

(σ is a most general unifier of B and D.)

First, rename variables apart in the clauses! For example, given

$$\{P(x)\} \quad \text{and} \quad \{\neg P(g(x))\},$$

we must rename x in one of the clauses. (Otherwise, unification fails.)

The Factoring Rule

This inference collapses unifiable literals in one clause:

$$\frac{\{B_1, \ldots, B_k, A_1, \ldots, A_m\}}{\{B_1, A_1, \ldots, A_m\}\sigma} \qquad \text{provided } B_1\sigma = \cdots = B_k\sigma$$

Example: Prove $\forall x\, \exists y\, \neg(P(y,x) \leftrightarrow \neg P(y,y))$

The clauses are	$\{\neg P(y,a), \neg P(y,y)\}$	$\{P(y,y), P(y,a)\}$
Factoring yields	$\{\neg P(a,a)\}$	$\{P(a,a)\}$

Resolution yields the empty clause!

A Non-Trivial Proof

$$\exists x\,[P \to Q(x)] \land \exists x\,[Q(x) \to P] \to \exists x\,[P \leftrightarrow Q(x)]$$

Clauses are $\{P, \neg Q(b)\}$ $\{P, Q(x)\}$ $\{\neg P, \neg Q(x)\}$ $\{\neg P, Q(a)\}$

Resolve $\{P, \underline{\neg Q(b)}\}$ with $\{P, \underline{Q(x)}\}$ getting $\{P, P\}$

Factor $\{P, P\}$ getting $\{P\}$

Resolve $\{\neg P, \underline{\neg Q(x)}\}$ with $\{\neg P, \underline{Q(a)}\}$ getting $\{\neg P, \neg P\}$

Factor $\{\neg P, \neg P\}$ getting $\{\neg P\}$

Resolve $\{P\}$ with $\{\neg P\}$ getting $\square$

What About Equality?

In theory, it's enough to add the equality axioms:

- The reflexive, symmetric and transitive laws.
- Substitution laws like $\{x \neq y, f(x) = f(y)\}$ for each f.
- Substitution laws like $\{x \neq y, \neg P(x), P(y)\}$ for each P.

In practice, we need something special: the paramodulation rule

$$\frac{\{B[t'], A_1, \ldots, A_m\} \qquad \{t = u, C_1, \ldots, C_n\}}{\{B[u], A_1, \ldots, A_m, C_1, \ldots, C_n\}\sigma} \qquad \text{(if } t\sigma = t'\sigma\text{)}$$

Prolog Clauses

Prolog clauses have a restricted form, with at most one positive literal.

The definite clauses form the program. Procedure B with body "commands" $A_1, \ldots, A_m$ is

$$B \leftarrow A_1, \ldots, A_m$$

The single goal clause is like the "execution stack", with say m tasks left to be done.

$$\leftarrow A_1, \ldots, A_m$$

Prolog Execution

Linear resolution:

- Always resolve some program clause with the goal clause.
- The result becomes the new goal clause.

Try the program clauses in left-to-right order.

Solve the goal clause's literals in left-to-right order.

Use depth-first search. (Performs backtracking, using little space.)

Do unification without occurs check. (Unsound, but needed for speed)

A (Pure) Prolog Program

```
parent(elizabeth,charles).
parent(elizabeth,andrew).

parent(charles,william).
parent(charles,henry).

parent(andrew,beatrice).
parent(andrew,eugenia).

grand(X,Z) :- parent(X,Y), parent(Y,Z).
cousin(X,Y) :- grand(Z,X), grand(Z,Y).
```

Prolog Execution

```
                                         :- cousin(X,Y).
                              :- grand(Z1,X), grand(Z1,Y).
           :- parent(Z1,Y2), parent(Y2,X), grand(Z1,Y).
*            :- parent(charles,X), grand(elizabeth,Y).
X=william                       :- grand(elizabeth,Y).
               :- parent(elizabeth,Y5), parent(Y5,Y).
*                                 :- parent(andrew,Y).
Y=beatrice                                         :- □.
```

$\star$ = backtracking choice point

16 solutions including `cousin(william,william)`

and `cousin(william,henry)`

Another FOL Proof Procedure: Model Elimination

A Prolog-like method to run on fast Prolog architectures.

Contrapositives: treat clause $\{A_1, \ldots, A_m\}$ like the m clauses

$$A_1 \leftarrow \neg A_2, \ldots, \neg A_m$$

$$A_2 \leftarrow \neg A_3, \ldots, \neg A_m, \neg A_1$$

$$\vdots$$

$$A_m \leftarrow \neg A_1, \ldots, \neg A_{m-1}$$

Extension rule: when proving goal P, assume $\neg P$.

A Survey of Automatic Theorem Provers

First-order Resolution: E, SPASS, Vampire, . . .

Higher-Order Logic: TPS, LEO and LEO-II, Satallax

Model Elimination: Prolog Technology Theorem Prover, SETHEO (historical)

Parallel ME: PARTHENON, PARTHEO

Tableau (sequent) based: LeanTAP, 3TAP, . . .

Decision Problems

Any formally-stated question: is n prime or not? Is the string s accepted by a given context-free grammar?

Unfortunately, most decision problems for logic are difficult:

- Propositional satisfiability NP-complete.
- The halting problem is undecidable. Therefore there is no decision procedure to identify first-order theorems.
- The theory of integer arithmetic is undecidable (Gödel).

Solvable Decision Problems

Propositional formulas are decidable: use the DPLL algorithm.

Linear arithmetic formulas are decidable:

- comparisons using $+$ and $-$ but $\times$ only with constants, e.g.
- $2x < y \land y < x$ (satisfiable by $y = -3$, $x = -2$) or
  $2x < y \land y < x \land 3x > 2$ (unsatisfiable)
- the integer and real (or rational) cases require different algorithms

Polynomial arithmetic is decidable, and so is Euclidean geometry.

Fourier-Motzkin Variable Elimination

Decides conjunctions of linear constraints over reals/rationals

$$\bigwedge_{i=1}^{m} \sum_{j=1}^{n} a_{ij}x_j \leq b_i$$

Eliminate variables one-by-one until one remains, or contradiction

Devised by Fourier (1826) — resembles Gaussian elimination

One of the first decision procedures to be implemented

Worst-case complexity: $O(m^{2^n})$

Basic Idea: Upper and Lower Bounds

To eliminate variable x_n, consider constraint i, for $i = 1, \ldots, m$:

Define $\beta_i = b_i - \sum_{j=1}^{n-1} a_{ij} x_j$. Rewrite constraint i:

$$\text{If } a_{in} > 0 \text{ then } x_n \leq \frac{\beta_i}{a_{in}}$$

$$\text{if } a_{in} < 0 \text{ then } -x_n \leq -\frac{\beta_i}{a_{in}}$$

Adding two such constraints yields $0 \leq \frac{\beta_i}{a_{in}} - \frac{\beta_{i'}}{a_{i'n}}$

Do this for all combinations with opposite signs

Then delete original constraints (except where $a_{in} = 0$)

Fourier-Motzkin Elimination Example

initial problem	eliminate x	eliminate z	result
$x \leq y$	$z \leq 0$	$0 \leq -1$	UNSAT
$x \leq z$	$y + z \leq 0$	$y \leq -1$	
$-x + y + 2z \leq 0$			
$-z \leq -1$	$-z \leq -1$		

Quantifier Elimination (QE)

Skolemization eliminates quantifiers but only preserves consistency.

QE transforms a formula to a quantifier-free but equivalent formula.

The idea of Fourier-Motzkin is that (e.g.)

$$\exists xy\,(2x < y \land y < x) \iff \exists x\, 2x < x \iff \mathbf{t}$$

In general, the quantifier-free formula is **enormous**.

- With no free variables, the end result must be **t** or **f**.
- But even then, the time complexity tends to be hyper-exponential!

Other Decidable Theories

Linear integer arithmetic: use Omega test or Cooper's algorithm, but any decision algorithm has a worst-case runtime of at least $2^{2^{cn}}$

QE for real polynomial arithmetic:

$$\exists x\, [ax^2 + bx + c = 0] \iff b^2 \geq 4ac \land (c = 0 \lor a \neq 0 \lor b^2 > 4ac)$$

There exist decision procedures for arrays, lists, bit vectors, ...

Sometimes, they can cooperate to decide combinations of theories.

Problem: To Combine Theories with Boolean Logic

These procedures expect existentially quantified conjunctions.

Formulas must be converted to disjunctive normal form.

Universal quantifiers must be eliminated using $\forall x\, A \simeq \neg(\exists x\, (\neg A))$.

Could there be a better way? Couldn't we somehow use DPLL?

Satisfiability Modulo Theories

Idea: use DPLL for logical reasoning, decision procedures for theories

Clauses can have literals like $2x < y$, which are used as names.

If DPLL finds a contradiction, then the clauses are unsatisfiable.

Asserted literals are checked by the decision procedure:

- Unsatisfiable conjunctions of literals are noted as new clauses.
- Case splitting is interleaved with decision procedure calls.

SMT Example

$$\frac{\{c=0, 2a<b\}\ \{b<a\}\ \{3a>2, a<0\}\ \{c\neq 0, \neg(b<a)\}}{}$$

$\{c=0, 2a<b\}$	$\{3a>2, a<0\}$	$\{c\neq 0\}$	unit $b<a$
$\{2a<b\}$	$\{3a>2, a<0\}$		unit $c\neq 0$
	$\{3a>2, a<0\}$		unit $2a<b$

Now a case split returns a "model": $b<a, c\neq 0, 2a<b, 3a>2$

But the dec. proc. finds these contradictory and returns a new clause:

$$\{\neg(b<a), \neg(2a<b), \neg(3a>2)\}$$

Finally get a satisfiable result: $b<a \land c\neq 0 \land 2a<b \land a<0$

Remarks on the Previous Example

DPLL works only for propositional formulas!

We should properly write

$$\boxed{c = 0}, \boxed{2a < b}\} \quad \{\neg \boxed{c = 0}, \neg \boxed{b < a}\} \quad \text{etc.}$$

The DPLL part knows nothing about arithmetic.

SMT makes two independent reasoners cooperate!

SMT Solvers and Their Applications

Popular ones include Z3, Yices, CVC4, but there are many others.

Representative applications:

- Hardware and software verification
- Program analysis and symbolic software execution
- Planning and constraint solving
- Hybrid systems and control engineering

BDDs: Binary Decision Diagrams

A canonical form for boolean expressions: decision trees with sharing.

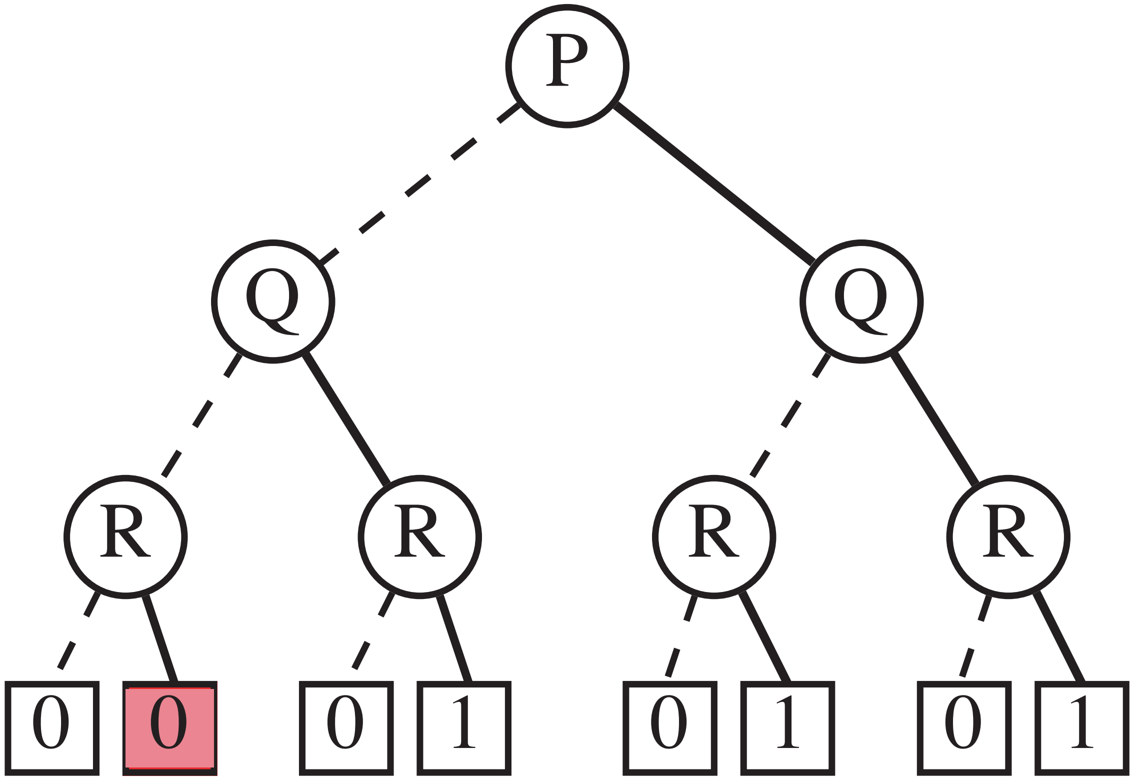
- ordered propositional symbols (the variables)
- sharing of identical subtrees
- hashing and other optimisations

Detects if a formula is tautologous (=1) or inconsistent (=0).

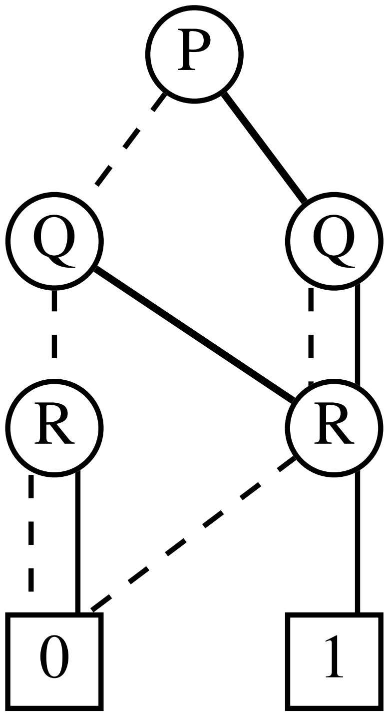
Exhibits models (paths to 1) if the formula is satisfiable.

Excellent for verifying digital circuits, with many other applications.

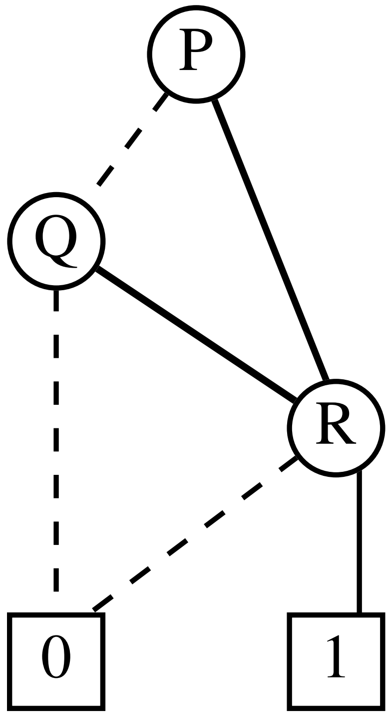
Decision Diagram for $(P \vee Q) \wedge R$



Converting a Decision Diagram to a BDD



No duplicates



No redundant tests

Building BDDs Efficiently

Do not construct the full binary tree!

Do not expand $\rightarrow$, $\leftrightarrow$, $\oplus$ (exclusive OR) to other connectives!!

- Recursively convert operands to BDDs.
- Combine operand BDDs, respecting the ordering and sharing.
- Delete redundant variable tests.

Canonical Form Algorithm

To convert $Z \land Z'$, where Z and Z' are already BDDs:

Trivial if either operand is 1 or 0.

Let $Z = \textbf{if}(P, X, Y)$ and $Z' = \textbf{if}(P', X', Y')$

- If $P = P'$ then recursively convert $\textbf{if}(P,\ X \land X',\ Y \land Y')$.
- If $P < P'$ then recursively convert $\textbf{if}(P,\ X \land Z',\ Y \land Z')$.
- If $P > P'$ then recursively convert $\textbf{if}(P',\ Z \land X',\ Z \land Y')$.

Canonical Forms of Other Connectives

$Z \lor Z'$, $Z \to Z'$ and $Z \leftrightarrow Z'$ are converted to BDDs similarly.

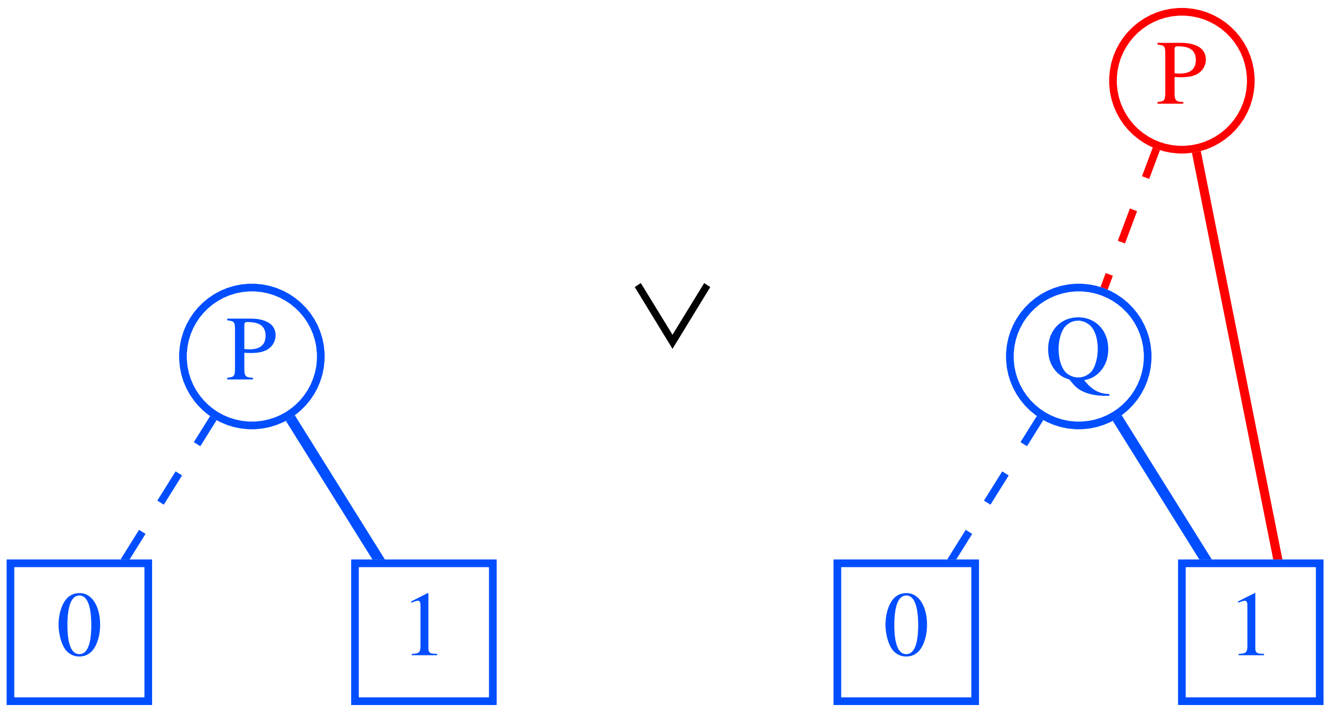
Some cases, like $Z \to 0$ and $Z \leftrightarrow 0$, reduce to negation.

Here is how to convert $\neg Z$, where Z is a BDD:

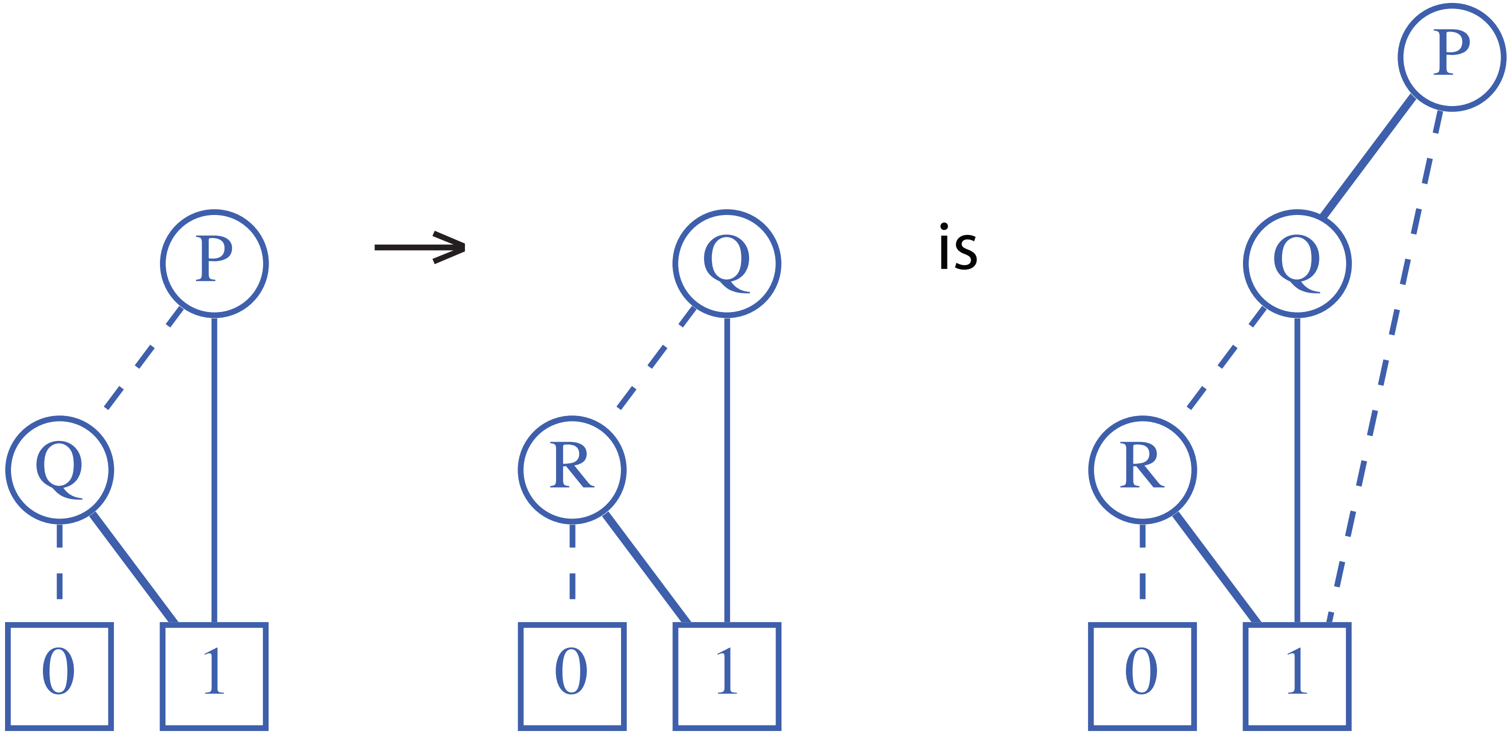
- If $Z = \textbf{if}(P, X, Y)$ then recursively convert $\textbf{if}(P, \neg X, \neg Y)$.
- if $Z = 1$ then return 0, and if $Z = 0$ then return 1.

(In effect we copy the BDD but exchange the 1 and 0 at the bottom.)

Canonical Form (that is, BDD) of $P \vee Q$



Canonical Form of $P \vee Q \to Q \vee R$



Optimisations

Never build the same BDD twice, but share pointers. Advantages:

- If $X \simeq Y$, then the addresses of X and Y are equal.
- Can see if **if**(P, X, Y) is redundant by checking if $X = Y$.
- Can quickly simplify special cases like $X \wedge X$.

Never convert $X \wedge Y$ twice, but keep a hash table of known canonical forms. This prevents redundant computations.

Final Observations

The variable ordering is crucial. Consider this formula:

$$(P_1 \land Q_1) \lor \cdots \lor (P_n \land Q_n)$$

A good ordering is $P_1 < Q_1 < \cdots < P_n < Q_n$: the BDD is linear.

With $P_1 < \cdots < P_n < Q_1 < \cdots < Q_n$, the BDD is exponential.

Many digital circuits have small BDDs: adders, but not multipliers.

BDDs can solve problems in hundreds of variables.

The general case remains hard (it is NP-complete).

Modal Operators

W: set of possible worlds (machine states, future times, . . .)

R: accessibility relation between worlds

(W, R) is called a modal frame

$$\left.\begin{array}{l}\Box A \text{ means } A \text{ is necessarily true} \\ \Diamond A \text{ means } A \text{ is possibly true}\end{array}\right\} \text{ in all worlds accessible from here}$$

$\neg\Diamond A \simeq \Box\neg A$ $\qquad$ A cannot be true $\iff$ A must be false

Semantics of Propositional Modal Logic

For a particular frame (W, R)

An interpretation I maps the propositional letters to subsets of W

$w \Vdash A$ means A is true in world w

$$
\begin{aligned}
w \Vdash P &\iff w \in I(P) \\
w \Vdash A \land B &\iff w \Vdash A \text{ and } w \Vdash B \\
w \Vdash \Box A &\iff v \Vdash A \text{ for all } v \text{ such that } R(w, v) \\
w \Vdash \Diamond A &\iff v \Vdash A \text{ for some } v \text{ such that } R(w, v)
\end{aligned}
$$

Truth and Validity in Modal Logic

For a particular frame (W, R), and interpretation I

$w \Vdash A$	means A is true in world w
$\models_{W,R,I} A$	means $w \Vdash A$ for all w in W
$\models_{W,R} A$	means $w \Vdash A$ for all w and all I

$\models A$ means $\models_{W,R} A$ for all frames; A is universally valid

. . . but typically we constrain R to be, say, transitive.

All propositional tautologies are universally valid!

A Hilbert-Style Proof System for K

Extend your favourite propositional proof system with

$$\text{Dist} \quad \Box(A \to B) \to (\Box A \to \Box B)$$

Inference Rule: Necessitation

$$\frac{A}{\Box A}$$

Treat $\Diamond$ as a definition

$$\Diamond A \stackrel{\text{def}}{=} \neg\Box\neg A$$

Variant Modal Logics

Start with pure modal logic, which is called K

Add axioms to constrain the accessibility relation:

T	$\Box A \to A$	(reflexive)	logic T
4	$\Box A \to \Box\Box A$	(transitive)	logic $S4$
B	$A \to \Box\Diamond A$	(symmetric)	logic $S5$

And countless others!

We mainly look at $S4$, which resembles a logic of time.

Extra Sequent Calculus Rules for $S4$

$$\frac{A, \Gamma \Rightarrow \Delta}{\Box A, \Gamma \Rightarrow \Delta}\ (\Box l) \qquad \frac{\Gamma^* \Rightarrow \Delta^*, A}{\Gamma \Rightarrow \Delta, \Box A}\ (\Box r)$$

$$\frac{A, \Gamma^* \Rightarrow \Delta^*}{\Diamond A, \Gamma \Rightarrow \Delta}\ (\Diamond l) \qquad \frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, \Diamond A}\ (\Diamond r)$$

$\Gamma^* \stackrel{\text{def}}{=} \{\Box B \mid \Box B \in \Gamma\}$ Erase non-$\Box$ assumptions.

$\Delta^* \stackrel{\text{def}}{=} \{\Diamond B \mid \Diamond B \in \Delta\}$ Erase non-$\Diamond$ goals!

A Proof of the Distribution Axiom

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{A \Rightarrow B, A} \qquad \overline{B, A \Rightarrow B}}{A \to B, A \Rightarrow B}\,(\to l)}{A \to B, \Box A \Rightarrow B}\,(\Box l)}{\Box(A \to B), \Box A \Rightarrow B}\,(\Box l)}{\Box(A \to B), \Box A \Rightarrow \Box B}\,(\Box r)$$

And thus $\Box(A \to B) \to (\Box A \to \Box B)$

Must apply $(\Box r)$ first!

Part of an "Operator String Equivalence"

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{\Diamond A \Rightarrow \Diamond A}}{\Box\Diamond A \Rightarrow \Diamond A}\,(\Box l)}{\Diamond\Box\Diamond A \Rightarrow \Diamond A}\,(\Diamond l)}{\Box\Diamond\Box\Diamond A \Rightarrow \Diamond A}\,(\Box l)}{\Box\Diamond\Box\Diamond A \Rightarrow \Box\Diamond A}\,(\Box r)$$

In fact, $\Box\Diamond\Box\Diamond A \simeq \Box\Diamond A$ also $\Box\Box A \simeq \Box A$

The $S4$ operator strings are $\Box$ $\Diamond$ $\Box\Diamond$ $\Diamond\Box$ $\Box\Diamond\Box$ $\Diamond\Box\Diamond$

Two Failed Proofs

$$\dfrac{\dfrac{\Rightarrow A}{\Rightarrow \Diamond A}\;(\Diamond r)}{A \Rightarrow \Box\Diamond A}\;(\Box r)$$

$$\dfrac{\dfrac{B \Rightarrow A \land B}{B \Rightarrow \Diamond(A \land B)}\;(\Diamond r)}{\Diamond A, \Diamond B \Rightarrow \Diamond(A \land B)}\;(\Diamond l)$$

Can extract a countermodel from the proof attempt

Simplifying the Sequent Calculus

7 connectives (or 9 for modal logic):

$$\neg \quad \wedge \quad \vee \quad \rightarrow \quad \leftrightarrow \quad \forall \quad \exists \qquad (\Box \quad \Diamond)$$

Left and right: so 14 rules (or 18) plus basic sequent, cut

Idea! Work in Negation Normal Form

Fewer connectives: $\wedge \quad \vee \quad \forall \quad \exists \qquad (\Box \quad \Diamond)$

Sequents need one side only!

Tableau Calculus: Left-Only

$$\frac{}{\neg A, A, \Gamma \Rightarrow}\ \text{(basic)} \qquad \frac{\neg A, \Gamma \Rightarrow \qquad A, \Gamma \Rightarrow}{\Gamma \Rightarrow}\ \text{(cut)}$$

$$\frac{A, B, \Gamma \Rightarrow}{A \land B, \Gamma \Rightarrow}\ (\land l) \qquad \frac{A, \Gamma \Rightarrow \qquad B, \Gamma \Rightarrow}{A \lor B, \Gamma \Rightarrow}\ (\lor l)$$

$$\frac{A[t/x], \Gamma \Rightarrow}{\forall x\, A, \Gamma \Rightarrow}\ (\forall l) \qquad \frac{A, \Gamma \Rightarrow}{\exists x\, A, \Gamma \Rightarrow}\ (\exists l)$$

Rule $(\exists l)$ holds provided x is not free in the conclusion!

Tableau Rules for $S4$

$$\frac{A, \Gamma \Rightarrow}{\Box A, \Gamma \Rightarrow} \; (\Box l) \qquad \frac{A, \Gamma^* \Rightarrow}{\Diamond A, \Gamma \Rightarrow} \; (\Diamond l)$$

$$\Gamma^* \stackrel{\text{def}}{=} \{\Box B \mid \Box B \in \Gamma\}$$ Erase non-$\Box$ assumptions

From 14 (or 18) rules to 4 (or 6)

Left-side only system uses proof by contradiction

Right-side only system is an exact dual

Tableau Proof of $\forall x\,(P \to Q(x)) \to [P \to \forall y\, Q(y)]$

Negate and convert to NNF:

$$P,\ \exists y\, \neg Q(y),\ \forall x\,(\neg P \vee Q(x)) \Rightarrow$$

$$\dfrac{\dfrac{\dfrac{\overline{P,\ \neg Q(y),\ \neg P \Rightarrow} \qquad \overline{P,\ \neg Q(y),\ Q(y) \Rightarrow}}{P,\ \neg Q(y),\ \neg P \vee Q(y) \Rightarrow}\,(\vee l)}{P,\ \neg Q(y),\ \forall x\,(\neg P \vee Q(x)) \Rightarrow}\,(\forall l)}{P,\ \exists y\, \neg Q(y),\ \forall x\,(\neg P \vee Q(x)) \Rightarrow}\,(\exists l)$$

The Free-Variable Tableau Calculus

Rule $(\forall l)$ now inserts a new free variable:

$$\frac{A[z/x], \Gamma \Rightarrow}{\forall x\, A, \Gamma \Rightarrow} \ (\forall l)$$

Let unification instantiate any free variable

In $\neg A, B, \Gamma \Rightarrow$ try unifying A with B to make a basic sequent

Updating a variable affects entire proof tree

What about rule $(\exists l)$? Do not use it! Instead, Skolemize!

Skolemization from NNF

Recall e.g. that we Skolemize

$$[\forall y\, \exists z\, Q(y, z)] \land \exists x\, P(x) \quad \text{to} \quad [\forall y\, Q(y, f(y))] \land P(a)$$

Remark: pushing quantifiers in (miniscoping) gives better results.

Example: proving $\exists x\, \forall y\, [P(x) \to P(y)]$:

Negate; convert to NNF:	$\forall x\, \exists y\, [P(x) \land \neg P(y)]$
Push in the $\exists y$:	$\forall x\, [P(x) \land \exists y\, \neg P(y)]$
Push in the $\forall x$:	$(\forall x\, P(x)) \land (\exists y\, \neg P(y))$
Skolemize:	$\forall x\, P(x) \land \neg P(a)$

Free-Variable Tableau Proof of $\exists x\, \forall y\, [P(x) \to P(y)]$

$$
\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{y \mapsto f(z)}{P(y),\ \neg P(f(y)),\ P(z),\ \neg P(f(z)) \Rightarrow}\ \text{(basic)}}{P(y),\ \neg P(f(y)),\ P(z) \land \neg P(f(z)) \Rightarrow}\ (\land l)}{P(y),\ \neg P(f(y)),\ \forall x\, [P(x) \land \neg P(f(x))] \Rightarrow}\ (\forall l)}{P(y) \land \neg P(f(y)),\ \forall x\, [P(x) \land \neg P(f(x))] \Rightarrow}\ (\land l)}{\forall x\, [P(x) \land \neg P(f(x))] \Rightarrow}\ (\forall l)
$$

Unification chooses the term for $(\forall l)$

A Failed Proof

Try to prove $\forall x\,[P(x) \lor Q(x)] \to [\forall x\, P(x) \lor \forall x\, Q(x)]$

NNF: $\exists x\, \neg P(x) \land \exists x\, \neg Q(x) \land \forall x\, [P(x) \lor Q(x)] \Rightarrow$

Skolemize: $\neg P(a),\ \neg Q(b),\ \forall x\, [P(x) \lor Q(x)] \Rightarrow$

$$\dfrac{\dfrac{\dfrac{y \mapsto a}{\neg P(a),\ \neg Q(b),\ P(y) \Rightarrow} \qquad \dfrac{y \mapsto b???}{\neg P(a),\ \neg Q(b),\ Q(y) \Rightarrow}}{\neg P(a),\ \neg Q(b),\ P(y) \lor Q(y) \Rightarrow}\,(\lor l)}{\neg P(a),\ \neg Q(b),\ \forall x\, [P(x) \lor Q(x)] \Rightarrow}\,(\forall l)$$

The World's Smallest Theorem Prover?

```
prove((A,B),UnExp,Lits,FreeV,VarLim) :- !,                    and
        prove(A,[B|UnExp],Lits,FreeV,VarLim).
prove((A;B),UnExp,Lits,FreeV,VarLim) :- !,                    or
        prove(A,UnExp,Lits,FreeV,VarLim),
        prove(B,UnExp,Lits,FreeV,VarLim).
prove(all(X,Fml),UnExp,Lits,FreeV,VarLim) :- !,               forall
        \+ length(FreeV,VarLim),
        copy_term((X,Fml,FreeV),(X1,Fml1,FreeV)),
        append(UnExp,[all(X,Fml)],UnExp1),
        prove(Fml1,UnExp1,Lits,[X1|FreeV],VarLim).
prove(Lit,_,[L|Lits],_,_) :-                                  literals; negation
        (Lit = -Neg; -Lit = Neg) ->
        (unify(Neg,L); prove(Lit,[],Lits,_,_)).
prove(Lit,[Next|UnExp],Lits,FreeV,VarLim) :-                  next formula
        prove(Next,UnExp,[Lit|Lits],FreeV,VarLim).
```