# Basic set theory and logic

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The 'Introduction to NLP' module assumes some basic knowledge of set theory and logic. Students need to be familiar with the following concepts:

1. The basic idea of a set, set intersection, set union etc. Venn diagrams.
2. Propositional logic, interpretation with respect to a model (expressed in terms of sets).
3. Truth tables. Tautology and contradiction.
4. First order predicate calculus, universal and existential quantifiers.

They should also have some understanding of how these concepts apply to the analysis of natural language.

This document contains some exercises which should serve to indicate the background that is assumed. Students who have difficulty with the exercises should work though an introductory logic book: suggestions for reading are given at the end of the document. I am happy to check answers if you want to hand them in.

## 1 Notation

Unfortunately, logical notation is not standardized. The following notation is assumed in the examples and in my lectures:

### Set theory

| | |
|---|---|
| $\{a, b, c\}$ | the set containing a, b and c |
| $\in$ | set membership |
| $\emptyset$ | the empty set |
| $\subset$ | proper subset (i.e., not =) |
| $\subseteq$ | subset |
| $=$ | equality |
| $\cap$ | intersection |
| $\cup$ | union |
| $-$ | set difference |

### Logic

| | |
|---|---|
| $\neg$ | negation |
| $\wedge$ | conjunction |
| $\vee$ | disjunction |
| $\Rightarrow$ | implication |
| $\iff$ | mutual implication |
| $\forall$ | universal quantifier |
| $\exists$ | existential quantifier |

# 2 Exercises

## 2.1 Set theory and logic

1. Draw shaded set diagrams (Venn diagrams) corresponding to:
   (a) $A \cap (B \cup C)$
   (b) $A - B$ (i.e., set difference)
2. Draw set diagrams corresponding to the following logical expressions:
   (a) $\neg P \vee Q$
   (b) $\neg(P \wedge Q) \wedge R$
3. The sentence *every cat does not sleep* has two possible interpretations:
   (a) No cat sleeps
   (b) It is not the case that every cat sleeps

   Draw the Venn diagrams for these two interpretations, and give a specific model (i.e., statement about what is true in some world) such that b) is true but a) isn't. Is it possible for a) to be true without b) being true?

## 2.2 Propositional logic and quantifier-free predicate logic

1. Draw the truth table for $P \Rightarrow Q$
2. Which of the following expressions are tautologies or contradictions? (Show the truth tables)
   (a) $P \Rightarrow \neg P$
   (b) $\neg(P \Rightarrow (P \vee Q))$
   (c) $\neg(P \vee Q) \Rightarrow P$
3. Assume that:

   Rover barks and it-is-not-the-case-that Kitty chases Rover

   corresponds to the logical expression:

   $\text{bark}'(r) \wedge \neg(\text{chase}'(k, r))$

   where $k$ and $r$ are constants corresponding to Kitty and Rover. We use the prime symbol (e.g.,bark′) to indicate we are talking about the predicate and not the word. A logical expression capturing the meaning of a sentence (approximately) is called the logical form for the sentence. I have used the artificial word 'it-is-not-the-case-that' to make the meaning of the sentence clearer for this example. We will also assume for the sake of the exercise that the logical connectives correspond to their English counterparts.

   Show a logical form or forms for each of the following sentences:

   (a) it-is-not-the-case-that Kitty chases Rover or Rover sleeps
   (b) if Lynx sleeps then Rover barks or Rover sleeps

   The sentences may be ambiguous, in which case you should give more than one logical form.
4. The truth of a logical form can be evaluated with respect to a model (i.e., statement of what is true in some world). For instance, in the question above:

   $\text{bark}'(r) \wedge \neg(\text{chase}'(k, r))$

could be evaluated with respect to the following model:

chase′: $\{\langle l, k\rangle, \langle l, l\rangle, \langle l, r\rangle\}$
sleep′: $\{\}$
bark′: $\{r\}$

In this case, bark′$(r)$ is true, and chase′$(k, r)$ is not true, so the expression as a whole is true.

Evaluate the truth of each logical form that you gave in the answer to the previous question with respect to this model.

## 2.3 First order predicate calculus (FOPC)

1. For each of the following expressions, underline the scope of each quantifier, and mark any free variables:

   (a) $\forall x[P(x) \Rightarrow [Q(x) \wedge P(y)]]$

   (b) $\exists z[\forall x[\exists y[P(x) \wedge Q(y) \Rightarrow \neg S(x, y)] \wedge P(x)] \wedge \neg[R(z) \Rightarrow Q(y)]]$

2. Consider the following model (where s is Sandy, k is Kim and l is Lee):

   like′: $\langle s, k\rangle, \langle l, k\rangle, \langle l, l\rangle, \langle s, s\rangle$
   student′: $\{s, k\}$
   beer-drinker′: $\{s, l\}$

   In this model, the following sentences are true (among others):

   (a) like′$(l, k)$
   Lee likes Kim

   (b) student′$(s)$
   Sandy is a student

   (c) $\neg\forall x$[student′$(x) \Rightarrow$ beer-drinker′$(x)$]
   It-is-not-the-case-that every student is a beer drinker.

   For each of the following, say whether it is true or false in the model, and give an English sentence that corresponds to the logical expression:

   (a) $\forall y[\forall x$[beer-drinker′$(x) \wedge$ beer-drinker′$(y) \Rightarrow$ like′$(x, y)$]]

   (b) $\exists z$[student′$(z) \wedge \forall y$[beer-drinker′$(y) \Rightarrow \neg$like′$(z, y)$]]

## 2.4 Translation into FOPC

English sentences can be (roughly) translated into FOPC (we will go over some ways this may be done automatically in the lectures, but for now we are just concerned with intuitively valid translations). For instance:

1. Every student drinks or smokes
   $\forall x$[student′$(x) \Rightarrow$ (drink′$(x) \vee$ smoke′$(x)$)]

2. Kim likes every student
   $\forall y$[student′$(y) \Rightarrow$ like′$(k, y)$]

3. Kim likes some students
   $\exists z$[student′$(z) \wedge$ like′$(k, z)$]

Give FOPC equivalents for the following sentences, making sure to give all possible equivalents in the case of ambiguity:

1. Not every student drinks
2. Every student does not drink
3. Every student likes Kim and some student does not drink

# 3 Reading

Jens Allwood, Lars-Gunnar Andersson, Östen Dahl, *Logic in Linguistics*, Cambridge University Press, 1977

A very good and quite gentle introduction geared towards the sort of logical concepts needed in computational linguistics.

Jon Barwise and John Etchemendy, *Language, Proof and Logic*, CSLI Publications, 2000

This is a book with an associated software package — it's not worth getting if you don't intend to use the software (Windows and Macs, not Linux). It is a very good way of learning logic really thoroughly, though it is not really oriented towards linguistics. The first two-thirds of the book is the part that is relevant for the course.

The following book goes deeper and covers some of the material in the lectures, although there is no book that covers the computational aspects of the course.

Ronnie Cann, *Formal semantics, an introduction*, Cambridge University Press, 1993

Recommended for coverage of logic, compositional semantics and lambda calculus. Covers a lot more formalism than we'll need.