# L11: Algebraic Path Problems with applications to Internet Routing Lecture 7 CAS Part III


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## Path Weight with functions on arcs?

For graph $G = (V,\ E)$, and arc path $p = (u_0,\ u_1)(u_1,\ u_2)\cdots(u_{k-1},\ u_k)$.

**Functions on arcs: two natural ways to do this...**

Weight function $w : E \to (S \to S)$. Let $f_j = w(u_{j-1}, u_j)$.

$$
\begin{aligned}
w_a^L(p) &= f_1(f_2(\cdots f_k(a)\cdots)) &= (f_1 \circ f_2 \circ \cdots \circ f_k)(a) \\
w_a^R(p) &= f_k(f_{k-1}(\cdots f_1(a)\cdots)) &= (f_k \circ f_{k-1} \circ \cdots \circ f_1)(a)
\end{aligned}
$$

How can we "make this work" for path problems?

# Algebra of Monoid Endomorphisms (AME) (See Gondran and Minoux 2008)

Let $(S, \oplus, \overline{0})$ be a commutative monoid.

$(S, \oplus, F \subseteq S \to S, \overline{0})$ is an algebra of monoid endomorphisms (AME) if

- $\forall f \in F, f(\overline{0}) = \overline{0}$
- $\forall f \in F, \forall b, c \in S, f(b \oplus c) = f(b) \oplus f(c)$

**I will declare these as optional**

- $\forall f, g \in F, f \circ g \in F$ (closed)
- $\exists i \in F, \forall s \in S, i(s) = s$
- $\exists \omega \in F, \forall n \in N, \omega(n) = \overline{0}$

Note: as with semirings, we may have to drop some of these axioms in order to model Internet routing ...



# So why do we want AMEs?

Each (closed with $\omega$ and $i$) AME can be viewed as a semiring of functions. Suppose $(S, \oplus, F, \overline{0})$ is an algebra of monoid endomorphisms. We can turn it into a semiring

$$\mathbb{F} = (F, \hat{\oplus}, \circ, \omega, i)$$

where $(f \ \hat{\oplus} \ g)(a) = f(a) \oplus g(a)$ and $(f \ \circ \ g)(a) = f(g(a))$.

**But functions are hard to work with....**

- All algorithms need to check equality over elements of a semiring
- $f = g$ means $\forall a \in S, f(a) = g(a)$
- $S$ can be very large, or infinite ....

# How do we represent a set of functions $F \subseteq S \to S$?

Assume we a set $\Lambda$ and a function

$$\rhd \in \Lambda \to (S \to S).$$

We normally write $\lambda \rhd s$ rather than $\rhd(\lambda)(s)$. We think of $\lambda \in \Lambda$ as the index for a function $f_\lambda(s) = \lambda \rhd s$. In this way $(\Lambda,\ \rhd)$ can be used to represent the set of functions

$$F = \{f_\lambda \mid \lambda \in \Lambda\}.$$



# Indexed Algebra of Monoid Endomorphisms (IAME)

Let $(S,\ \oplus,\ \overline{0})$ be a commutative and idempotent monoid.

**A (left) IAME $(S,\ \oplus,\ (\Lambda,\ \rhd),\ \overline{0})$**

- $\forall \lambda \in \Lambda,\ \lambda \rhd \overline{0} = \overline{0}$
- $\exists \lambda \in \Lambda,\ \forall s \in S,\ \lambda \rhd s = s$
- $\exists \lambda \in \Lambda,\ \forall s \in S,\ \lambda \rhd s = \overline{0}$
- $\forall \lambda \in \Lambda,\ \forall n, m \in S,\ \lambda \rhd (n \oplus m) = (\lambda \rhd n) \oplus (\lambda \rhd m)$

When we need closure? If needed, it would be

$$\forall \lambda_1, \lambda_2 \in \Lambda, \exists \lambda_3 \in \Lambda, \forall s \in S, \lambda_3 \rhd s = \lambda_1 \rhd (\lambda_2 \rhd s)$$

But this does not appear to be very useful ...

## IAME of Matrices

Given a left IAME $(S, \oplus, (L, \rhd), \overline{0})$ define the left IAME of matrices

$$(\mathbb{M}_n(S), \oplus, (\mathbb{M}_n(L), \rhd), \mathbf{J}).$$

For all $i, j$ we have $\mathbf{J}(i, j) = \overline{0}$. For $\mathbf{A} \in \mathbb{M}_n(L)$ and $\mathbf{B}, \mathbf{C} \in \mathbb{M}_n(S)$ define

$$(\mathbf{B} \oplus \mathbf{C})(i, j) \equiv \mathbf{B}(i, j) \oplus \mathbf{C}(i, j)$$

$$(\mathbf{A} \rhd \mathbf{B})(i, j) \equiv \bigoplus_{1 \leqslant q \leqslant n} \mathbf{A}(i, q) \rhd \mathbf{B}(q, j)$$



## Solving (some) equations. Left version here ...

We will be interested in solving for **L** equations of the form

$$\mathbf{L} = (\mathbf{A} \rhd \mathbf{L}) \oplus \mathbf{B}$$

Let

$$\begin{aligned} \mathbf{A} \rhd^0 \mathbf{B} &= \mathbf{B} \\ \mathbf{A} \rhd^{k+1} \mathbf{B} &= \mathbf{A} \rhd (\mathbf{A} \rhd^k \mathbf{B}) \end{aligned}$$

and

$$\mathbf{A} \rhd^{(k)} \mathbf{B} = \mathbf{A} \rhd^0 \mathbf{B} \oplus \mathbf{A} \rhd^1 \mathbf{B} \oplus \mathbf{A} \rhd^2 \mathbf{B} \oplus \cdots \oplus \mathbf{A} \rhd^k \mathbf{B}$$

$$\mathbf{A} \rhd^* \mathbf{B} = \mathbf{A} \rhd^0 \mathbf{B} \oplus \mathbf{A} \rhd^1 \mathbf{B} \oplus \mathbf{A} \rhd^2 \mathbf{B} \oplus \cdots \oplus \mathbf{A} \rhd^k \mathbf{B} \oplus \cdots$$

# Key result (again)

**$q$ stability**

If there exists a $q$ such that for all $\mathbf{B}$, $\mathbf{A} \rhd^{(q)} \mathbf{B} = \mathbf{A} \rhd^{(q+1)} \mathbf{B}$, then $\mathbf{A}$ is $q$-stable. Therefore, $\mathbf{A} \rhd^{*} \mathbf{B} = \mathbf{A} \rhd^{(q)} \mathbf{B}$.

**Theorm**

If $\mathbf{A}$ is $q$-stable, then $\mathbf{L} = \mathbf{A} \rhd^{*} (\mathbf{B})$ solves the equation

$$\mathbf{L} = (\mathbf{A} \rhd \mathbf{L}) \oplus \mathbf{B}.$$



# Example

$$\begin{array}{c}\text{TwoLevels}((S,\ \oplus_S,\ (\Lambda_S, \rhd_S)),\ (T,\ \oplus_T,\ (\Lambda_S, \rhd_S))) \\ \equiv \\ (S \uplus T,\ \oplus,\ (\Lambda_S \times \Lambda_T,\ \rhd))\end{array}$$

where

**$\oplus \equiv \oplus_S + \oplus_T$ over $S \uplus T$ is defined as**

$$\begin{array}{rcl}\text{inl}(s) \oplus \text{inl}(s') & \equiv & \text{inl}(s \oplus_S s') \\ \text{inr}(t) \oplus \text{inr}(t') & \equiv & \text{inr}(t \oplus_T t') \\ \text{inl}(s) \oplus \text{inr}(t) & \equiv & \text{inl}(s) \\ \text{inr}(t) \oplus \text{inl}(s) & \equiv & \text{inl}(s)\end{array}$$

**$\rhd \in (\Lambda_S \times \Lambda_T) \to ((S \uplus T) \to (S \uplus T))$ is defined as**

$$\begin{array}{rcl}(\lambda_S, \lambda_T) \rhd \text{inl}(s) & \equiv & \text{inl}(\lambda_S \rhd_S s) \\ (\lambda_S, \lambda_T) \rhd \text{inr}(t) & \equiv & \text{inr}(\lambda_T \rhd_T t)\end{array}$$

# Something familiar : Lexicographic product

$$(S,\ \oplus_S,\ (\Lambda_S,\ \rhd_S))\ \vec{\times}\ (T,\ \oplus_T,\ (\Lambda_T,\ \rhd_T))$$
$$\equiv$$
$$(S \times T,\ \oplus_S \vec{\times} \oplus_T,\ (\Lambda_S \times \Lambda_T,\ \rhd_S \times \rhd_T))$$

## Theorem

$$\mathbb{D}((S,\ \oplus_S,\ (\Lambda_S,\ \rhd_S))\ \vec{\times}\ (T,\ \oplus_T,\ (\Lambda_T,\ \rhd_T)))$$
$$\Longleftrightarrow$$
$$\mathbb{D}(S,\ \oplus_S,\ (\Lambda_S,\ \rhd_S)) \wedge \mathbb{D}(T,\ \oplus_T,\ (\Lambda_T,\ \rhd_T))$$
$$\wedge\ (\mathbb{C}(S,\ (\Lambda_S,\ \rhd_S)) \vee \mathbb{K}(T,\ (\Lambda_T,\ \rhd_T)))$$

Where

$$\begin{array}{rcl} \mathbb{D}(S,\ \oplus,\ (\Lambda,\ \rhd)) & \equiv & \forall a, b \in S,\ \lambda \in \Lambda,\ \lambda \rhd (a \oplus b) = (\lambda \rhd a) \oplus (\lambda \rhd b) \\ \mathbb{C}(S,\ (\Lambda,\ \rhd)) & \equiv & \forall a, b \in S,\ \lambda \in \Lambda,\ \lambda \rhd a = \lambda \rhd b \implies a = b \\ \mathbb{K}(S,\ (\Lambda,\ \rhd)) & \equiv & \forall a, b \in S,\ \lambda \in \Lambda,\ \lambda \rhd a = \lambda \rhd b \end{array}$$



# Something new: Functional Union

$$(S,\ \oplus,\ (\Lambda_1,\ \rhd_1)) +_{\mathrm{m}} (S,\ \oplus,\ (\Lambda_2,\ \rhd_2)) = (S,\ \oplus,\ (\Lambda_1 \uplus \Lambda_2,\ \rhd))$$

Where $\rhd \equiv \rhd_1 \uplus \rhd_2$ is defined as

$$\begin{array}{rcl} \mathrm{inl}(\lambda_1) \rhd s & = & \lambda \rhd_1 s \\ \mathrm{inr}(\lambda_2) \rhd s & = & \lambda \rhd_2 s \end{array}$$

## Fact

$$\mathbb{D}((S,\ \oplus,\ (\Lambda_1,\ \rhd_1)) +_{\mathrm{m}} (S,\ \oplus,\ (\Lambda_2,\ \rhd_2)))$$
$$\Longleftrightarrow$$
$$\mathbb{D}(S,\ \oplus,\ (\Lambda_1,\ \rhd_1)) \wedge \mathbb{D}(S,\ \oplus,\ (\Lambda_2,\ \rhd_2))$$

## Left and Right

**$\text{right}(S, \oplus) \equiv (S, \oplus, (\{R\}, \textbf{right}))$**

$$R \;\textbf{right}\; s = s$$

**$\text{left}(S, \oplus) \equiv (S, \oplus, (S, \textbf{left}))$**

$$s_1 \;\textbf{left}\; s_2 = s_1$$

**The following are always hold.**

$$\begin{array}{l} \mathbb{D}(\text{right}(S, \oplus)) \\ \mathbb{IP}(S, \oplus) \Rightarrow \mathbb{D}(\text{left}(S, \oplus)) \\ \mathbb{C}(\text{right}(S, \oplus)) \\ \mathbb{K}(\text{left}(S, \oplus)) \end{array}$$



## Scoped Product (Think iBGP/eBGP)

$$\begin{array}{c} (S, \oplus_S, (\Lambda_S, \rhd_S)) \ominus (T, \oplus_T, (\Lambda_T, \rhd_T)) \\ \equiv \\ ((S, \oplus_S, (\Lambda_S, \rhd_S)) \vec{\times} \text{left}(T, \oplus_T)) \\ +_{\text{m}} \\ (\text{right}(S, \oplus_S) \vec{\times} (T, \oplus_T, (\Lambda_T, \rhd_T))) \end{array}$$

**Between regions ($\lambda \in \Lambda_S$, $s \in S$, $t_1, t_2 \in T$)**

$$\text{inl}(\lambda, t_2) \rhd (s, t_1) = (\lambda \rhd_S s, t_2)$$

**Within regions ($\lambda \in \Lambda_T$, $s \in S$, $t \in T$)**

$$\text{inr}(R, \lambda) \rhd (s, t) = (s, \lambda \rhd_T t)$$

**Theorem.** If $\mathbb{IP}(T, \oplus_T)$, then

$$(\mathbb{D}((S, \oplus_S, (\Lambda_S, \rhd_S)) \ominus (T, \oplus_T, (\Lambda_T, \rhd_T)))$$
$$\Longleftrightarrow$$
$$\mathbb{D}(S, \oplus_S, (\Lambda_S, \rhd_S)) \wedge \mathbb{D}(T, \oplus_T, (\Lambda_T, \rhd_T)))$$

$$\begin{aligned}
& \mathbb{D}(((S, \oplus_S, (\Lambda_S, \rhd_S)) \vec{\times} \text{left}(T, \oplus_T)) \\
& \qquad +_{\text{m}} (\text{right}(S, \oplus_S) \vec{\times} (T, \oplus_T, (\Lambda_T, \rhd_T)))) \\
\Longleftrightarrow\ & \mathbb{D}((S, \oplus_S, (\Lambda_S, \rhd_S)) \vec{\times} \text{left}(T, \oplus_T)) \\
& \qquad \wedge\ \mathbb{D}((\text{right}(S, \oplus_S)) \vec{\times} (T, \oplus_T, (\Lambda_T, \rhd_T))) \\
\Longleftrightarrow\ & \mathbb{D}(S, \oplus_S, (\Lambda_S, \rhd_S)) \wedge \mathbb{D}(\text{left}(T, \oplus_T)) \\
& \qquad \wedge\ (\mathbb{C}(S, (\Lambda_S, \rhd_S)) \vee \mathbb{K}(\text{left}(T, \oplus_T))) \\
& \qquad \wedge\ \mathbb{D}(\text{right}(S, \oplus_S)) \wedge \mathbb{D}(T, \oplus_T, (\Lambda_T, \rhd_T)) \\
& \qquad \wedge\ (\mathbb{C}(\text{right}(S, \oplus_S)) \vee \mathbb{K}(T, (\Lambda_T, \rhd_T))) \\
\Longleftrightarrow\ & \mathbb{D}(S, \oplus_S, (\Lambda_S, \rhd_S)) \wedge \mathbb{D}(T, \oplus_T, (\Lambda_T, \rhd_T))
\end{aligned}$$