L11: Algebraic Path Problems with applications to Internet Routing Lecture 6 CAS Part II

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Distributivity?

Theorem: If $\oplus_S$ is commutative and selective, then

$$\mathbb{LD}((S, \oplus_S, \otimes_S) \vec{\times} (T, \oplus_T, \otimes_T)) \Leftrightarrow \mathbb{LD}(S, \oplus_S, \otimes_S) \wedge \mathbb{LD}(T, \oplus_T, \otimes_T) \wedge (\mathbb{LC}(S, \otimes_S) \vee \mathbb{LK}(T, \otimes_T))$$

$$\mathbb{RD}((S, \oplus_S, \otimes_S) \vec{\times} (T, \oplus_T, \otimes_T)) \Leftrightarrow \mathbb{RD}(S, \oplus_S, \otimes_S) \wedge \mathbb{RD}(T, \oplus_T, \otimes_T) \wedge (\mathbb{RC}(S, \otimes_S) \vee \mathbb{RK}(T, \otimes_T))$$

Left and Right Cancellative

$$\begin{array}{lcl} \mathbb{LC}(X, \bullet) & \equiv & \forall a, b, c \in X,\ c \bullet a = c \bullet b \Rightarrow a = b \\ \mathbb{RC}(X, \bullet) & \equiv & \forall a, b, c \in X,\ a \bullet c = b \bullet c \Rightarrow a = b \end{array}$$

Left and Right Constant

$$\begin{array}{lcl} \mathbb{LK}(X, \bullet) & \equiv & \forall a, b, c \in X,\ c \bullet a = c \bullet b \\ \mathbb{RK}(X, \bullet) & \equiv & \forall a, b, c \in X,\ a \bullet c = b \bullet c \end{array}$$

Why bisemigroups?

But wait! How could any semiring satisfy either of these properties?

$$\begin{array}{lcl} \mathbb{LC}(X, \bullet) & \equiv & \forall a, b, c \in X,\ c \bullet a = c \bullet b \Rightarrow a = b \\ \mathbb{LK}(X, \bullet) & \equiv & \forall a, b, c \in X,\ c \bullet a = c \bullet b \end{array}$$

- For $\mathbb{LC}$, note that we always have $\overline{0} \otimes a = \overline{0} \otimes b$, so $\mathbb{LC}$ could only hold when $S = \{\overline{0}\}$.
- For $\mathbb{LK}$, let $a = \overline{1}$ and $b = \overline{0}$ and $\mathbb{LK}$ leads to the conclusion that every c is equal to $\overline{0}$ (again!).

Normally we will add a zero and/or a one as the last step(s) of constructing a semiring. Alternatively, we might want to complicate our properties so that things work for semirings. A design trade-off!

Proof of $\Leftarrow$ for $\mathbb{LD}$ (Very carefully ...)

Assume

$$\begin{array}{ll}
(1) & \mathbb{LD}(S,\ \oplus_S,\ \otimes_S) \\
(2) & \mathbb{LD}(T,\ \oplus_T,\ \otimes_T) \\
(3) & \mathbb{LC}(S,\ \otimes_S) \vee \mathbb{LK}(T,\ \otimes_T) \\
(4) & \mathbb{IP}(S,\ \oplus_S).
\end{array}$$

Let $\oplus \equiv \oplus_S \vec{\times} \oplus_T$ and $\otimes \equiv \otimes_S \times \otimes_T$. Suppose

$$(s_1, t_1),\ (s_2, t_2),\ (s_3, t_3) \in S \times T.$$

We want to show that

$$\begin{array}{lll}
\text{lhs} & \equiv & (s_1,\ t_1) \otimes ((s_2,\ t_2) \oplus (s_3,\ t_3)) \\
 & = & ((s_1,\ t_1) \otimes (s_2,\ t_2)) \oplus ((s_1,\ t_1) \otimes (s_3,\ t_3)) \\
 & \equiv & \text{rhs}
\end{array}$$

Proof of $\Leftarrow$ for $\mathbb{LD}$

We have

$$\begin{array}{rcl}
\text{lhs} & \equiv & (s_1,\ t_1) \otimes ((s_2,\ t_2) \oplus (s_3,\ t_3)) \\
& = & (s_1,\ t_1) \otimes (s_2 \oplus_S s_3,\ t_{\text{lhs}}) \\
& = & (s_1 \otimes_S (s_2 \oplus_S s_3),\ t_1 \otimes_T t_{\text{lhs}})
\end{array}$$

$$\begin{array}{rcl}
\text{rhs} & \equiv & ((s_1,\ t_1) \otimes (s_2,\ t_2)) \oplus ((s_1,\ t_1) \otimes (s_3,\ t_3)) \\
& = & (s_1 \otimes_S s_2,\ t_1 \otimes_T t_2) \oplus (s_1 \otimes_S s_3,\ t_1 \otimes_T t_3) \\
& = & ((s_1 \otimes_S s_2) \oplus_S (s_1 \otimes_S s_3),\ t_{\text{rhs}}) \\
& =_{(1)} & (s_1 \otimes_S (s_2 \oplus_S s_3),\ t_{\text{rhs}})
\end{array}$$

where t_{lhs} and t_{rhs} are determined by the appropriate case in the definition of $\oplus$. Finally, note that

$$\text{lhs} = \text{rhs} \Leftrightarrow t_{\text{rhs}} = t_1 \otimes t_{\text{lhs}}.$$

Proof by cases on $s_2 \oplus_S s_3$

Case 1 : $s_2 = s_2 \oplus_S s_3 = s_3$. Then $t_{\text{lhs}} = t_2 \oplus_T t_3$ and

$$t_1 \otimes_T t_{\text{lhs}} = t_1 \otimes_T (t_2 \oplus_T t_3) =_{(2)} (t_1 \otimes_T t_2) \oplus_T (t_1 \otimes_T t_3).$$

Since $s_2 = s_3$ we have $s_1 \otimes_S s_2 = s_1 \otimes_S s_3$ and

$$s_1 \otimes_S s_2 =_{(4)} (s_1 \otimes_S s_2) \oplus_S (s_1 \otimes_S s_3) =_{(4)} s_1 \otimes_S s_3.$$

Therefore,

$$t_{\text{rhs}} = (t_1 \otimes_T t_2) \oplus (t_1 \otimes_T t_3) = t_1 \otimes_T t_{\text{lhs}}.$$

Case 2 : $s_2 = s_2 \oplus_S s_3 \neq s_3$. Then $t_{\text{lhs}} = t_2$ and

$$t_1 \otimes_T t_{\text{lhs}} = t_1 \otimes_T t_2.$$

Since $s_2 = s_2 \oplus_S s_3$ we have

$$s_1 \otimes_S s_2 = s_1 \otimes_S (s_2 \oplus_S s_3) =_{(1)} (s_1 \otimes_S s_2) \oplus_S (s_1 \otimes_S s_3).$$

Case 2.1 $s_1 \otimes_S s_2 \neq s_1 \otimes_S s_3$. Then $t_{\mathrm{rhs}} = t_1 \otimes_T t_2 = t_1 \otimes_T t_{\mathrm{lhs}}$.
Case 2.2 $s_1 \otimes_S s_2 = s_1 \otimes_S s_3$. Then

$$t_{\mathrm{rhs}} = (t_1 \otimes_T t_2) \oplus_T (t_1 \otimes_T t_3) =_{(2)} t_1 \otimes_T (t_2 \oplus_T t_3)$$

We need to consider two subcases.
Case 2.2.1: Assume $\mathbb{LC}(S, \otimes_S)$. But $s_1 \otimes_S s_2 = s_1 \otimes_S s_3 \Rightarrow s_2 = s_3$, which is a contradiction.
Case 2.2.2 : Assume $\mathbb{LK}(T, \otimes_T)$. In this case we know

$$\forall a, b \in X,\ t_1 \otimes_T a = t_1 \otimes_T b.$$

Letting $a = t_2 \oplus_T t_3$ and $b = t_2$ we have

$$t_{\mathrm{rhs}} = t_1 \otimes_T (t_2 \oplus_T t_3) = t_1 \otimes_T t_2 = t_1 \otimes_T t_{\mathrm{lhs}}.$$

Case 3 : $s_2 \neq s_2 \oplus_S s_3 = s_3$. Similar to Case 2.

Other direction, $\Rightarrow$ (Very carefully ...)

Prove this:

$$\neg\mathbb{LD}(S,\ \oplus_S,\ \otimes_S) \vee \neg\mathbb{LD}(T,\ \oplus_T,\ \otimes_T) \vee (\neg\mathbb{LC}(S,\ \otimes_S) \wedge \neg\mathbb{LK}(T,\ \otimes_T))$$
$$\Rightarrow\ \neg\mathbb{LD}((S,\ \oplus_S,\ \otimes_S) \vec{\times} (T,\ \oplus_T,\ \otimes_T)).$$

Case 1: $\neg\mathbb{LD}(S,\ \oplus_S,\ \otimes_S)$. That is

$$\exists a, b, c \in S,\ a \otimes_S (b \oplus_S c) \neq (a \otimes_S b) \oplus_S (a \otimes_S c).$$

Pick any $t \in T$. Then for some $t_1, t_2, t_3 \in T$ we have

$$\begin{array}{cl} & (a,\ t) \otimes ((b,\ t) \oplus (c,\ t)) \\ = & (a,\ t) \otimes (b \oplus_S c,\ t_1) \\ = & (a, \otimes_S (b \oplus_S c),\ t_2) \\ \neq & ((a \otimes_S b) \oplus_S (a \otimes_S c),\ t_3) \\ = & (a \otimes_S b,\ t \otimes_T t) \oplus (a \otimes_S c,\ t \otimes_T t) \\ = & ((a,\ t) \otimes (b,\ t)) \oplus ((a,\ t) \otimes (c,\ t)) \end{array}$$

Case 2: $\neg\mathbb{LD}(T,\ \oplus_T,\ \otimes_T)$. Similar.

Case 3: $(\neg\mathbb{LC}(S,\ \otimes_S) \wedge \neg\mathbb{LK}(T,\ \otimes_T))$. That is

$$\exists a, b, c \in S,\ c \otimes_S a = c \otimes_S b \wedge a \neq b$$

and

$$\exists x, y, z \in T,\ z \otimes_T x \neq z \otimes_T y.$$

Since $\oplus_S$ is selective and $a \neq b$, we have $a = a \oplus_S b$ or $b = a \oplus_S b$.
Case 3.1 : Assume $a = a \oplus_S b \neq b$.
Suppose that $t_1, t_2, t_3 \in T$. Then

$$\begin{array}{lcl} \text{lhs} & \equiv & (c,\ t_1) \otimes ((a,\ t_2) \oplus (b,\ t_3)) \\ & = & (c,\ t_1) \otimes (a,\ t_2) \\ & = & (c \otimes_S a,\ t_1 \otimes_T t_2) \end{array}$$

$$\begin{array}{lcl} \text{rhs} & \equiv & ((c,\ t_1) \otimes (a,\ t_2)) \oplus ((c,\ t_1) \otimes (b,\ t_3)) \\ & = & (c \otimes_S a,\ t_1 \otimes_T t_2) \oplus (c \otimes_S b,\ t_1 \otimes_T t_3) \\ & = & (c \otimes_S a,\ (t_1 \otimes_T t_2) \oplus_T (t_1 \otimes_T t_3)) \end{array}$$

Our job now is to select t_1, t_2, t_3 so that

$$t_{\text{lhs}} \equiv t_1 \otimes_T t_2 \neq (t_1 \otimes_T t_2) \oplus_T (t_1 \otimes_T t_3) \equiv t_{\text{rhs}}.$$

We don't have very much to work with! Only

$$\exists x, y, z \in T,\ z \otimes_T x \neq z \otimes_T y.$$

In addition, we can assume $\mathbb{LD}(T,\ \oplus_T,\ \otimes_T)$ (otherwise, use Case 2!), so

$$t_{\text{rhs}} = t_1 \otimes_T (t_2 \oplus_T t_3).$$

We need to select t_1, t_2, t_3 so that

$$t_{\text{lhs}} \equiv t_1 \otimes_T t_2 \neq t_1 \otimes_T (t_2 \oplus_T t_3) \equiv t_{\text{rhs}}.$$

Case 3.1.1: $z \otimes_T x = z \otimes_T (x \oplus_T y)$. Then letting $t_1 = z$, $t_2 = y$, and $t_3 = x$ we have

$$t_{\text{lhs}} = z \otimes_T y \neq z \otimes_T x = z \otimes_T (x \oplus_T y) = z \otimes_T (y \oplus_T x) = t_{\text{rhs}}.$$

Case 3.1.2: $z \otimes_T x \neq z \otimes_T (x \oplus_T y)$. Then letting $t_1 = z$, $t_2 = x$, and $t_3 = y$ we have

$$t_{\text{lhs}} = z \otimes_T x \neq z \otimes_T (x \oplus_T y) = t_{\text{rhs}}.$$

Case 3.2 : Assume $b = a \oplus_S b \neq a$.
Suppose that $t_1, t_2, t_3 \in T$. Then

$$\begin{array}{rcl} \text{lhs} & \equiv & (c,\ t_1) \otimes ((a,\ t_2) \oplus (b,\ t_3)) \\ & = & (c,\ t_1) \otimes (b,\ t_3) \\ & = & (c \otimes_S b,\ t_1 \otimes_T t_3) \end{array}$$

$$\begin{array}{rcl} \text{rhs} & \equiv & ((c,\ t_1) \otimes (a,\ t_2)) \oplus ((c,\ t_1) \otimes (b,\ t_3)) \\ & = & (c \otimes_S a,\ t_1 \otimes_T t_2) \oplus (c \otimes_S b,\ t_1 \otimes_T t_3) \\ & = & (c \otimes_S b,\ (t_1 \otimes_T t_2) \oplus_T (t_1 \otimes_T t_3)) \\ & = & (c \otimes_S b,\ t_1 \otimes_T (t_2 \oplus_T t_3)) \end{array}$$

We need to select t_1, t_2, t_3 so that

$$t_{\text{lhs}} \equiv t_1 \otimes_T t_3 \neq t_1 \otimes_T (t_2 \oplus_T t_3) \equiv t_{\text{rhs}}.$$

Case 3.2.1: $z \otimes_T x = z \otimes_T (x \oplus_T y)$. Then Then letting $t_1 = z$, $t_2 = x$, and $t_3 = y$ we have

$$t_{\text{lhs}} = z \otimes_T y \neq z \otimes_T x = z \otimes_T (x \oplus_T y) = t_{\text{rhs}}.$$

Case 3.2.2: $z \otimes_T x \neq z \otimes_T (x \oplus_T y)$. letting $t_1 = z$, $t_2 = y$, and $t_3 = x$ we have

$$t_{\text{lhs}} = z \otimes_T x \neq z \otimes_T (x \oplus_T y) = z \otimes_T (y \oplus_T x) = t_{\text{rhs}}.$$

□

Computing Counter Examples

Note that from (a, b, c) such $c \otimes_S a = c \otimes_S b \wedge a \neq b$ and (x, y, z) such that $z \otimes_T x \neq z \otimes_T y$ our proof computes a counter example to LD as

$$
\begin{array}{l}
\text{if } a = a \oplus_S b \\
\text{then if } z \otimes_T x = (z \otimes_T x) \oplus_T (z \otimes_T y) \\
\qquad \text{then } ((a,\ z),\ (b,\ y),\ (c,\ x)) \\
\qquad \text{else } ((a,\ z),\ (b,\ x),\ (c,\ y)) \\
\text{else if } z \otimes_T x = (z \otimes_T x) \oplus_T (z \otimes_T y) \\
\qquad \text{then } ((a,\ z),\ (b,\ x),\ (c,\ y)) \\
\qquad \text{else } ((a,\ z),\ (b,\ y),\ (c,\ x))
\end{array}
$$

Examples

True or counter example

name	S	$\oplus$	$\otimes$	$\mathbb{LD}$	$\mathbb{LC}(S, \otimes)$	$\mathbb{LK}(S, \otimes)$
min_plus	$\mathbb{N}$	min	$+$	$\star$	$\star$	$(0, 0, 1)$
max_min	$\mathbb{N}$	max	min	$\star$	$(0, 0, 1)$	$(1, 0, 1)$

For example, $(0, 0, 1)$ is a counter example for $\mathbb{LC}(\mathbb{N}, \text{min})$ since $0 \min 0 = 0 \min 1$, but $0 \neq 1$.

Let's turn the crank

$$
\begin{array}{ll}
 & \mathbb{LD}(\text{min\_plus} \vec{\times} \text{max\_min}) \\
\Leftrightarrow & \mathbb{LD}(\text{min\_plus}) \wedge \mathbb{LD}(\text{max\_min}) \wedge (\mathbb{LC}(\mathbb{N},\ +) \vee \mathbb{LK}(\mathbb{N},\ \text{min})) \\
\Leftrightarrow & \text{TRUE}
\end{array}
$$

Examples

Another turn of the crank

$$
\begin{array}{ll}
 & \mathbb{LD}(\text{max\_min} \ \vec{\times} \ \text{min\_plus}) \\
\Leftrightarrow & \mathbb{LD}(\text{max\_min}) \wedge \mathbb{LD}(\text{min\_plus}) \wedge (\mathbb{LC}(\mathbb{N}, \ \text{min}) \vee \mathbb{LK}(\mathbb{N}, \ +)) \\
\Leftrightarrow & \text{FALSE}
\end{array}
$$

Note that the counter examples to $\mathbb{LC}$ and $\mathbb{LK}$ can be plugged into the proof above to produce the a counter example to $\mathbb{LD}$,

$$((0, \ 0), \ (0, \ 0), \ (1, \ 1))$$

and sure enough, with $\oplus = \text{max} \ \vec{\times} \ \text{min}$ and $\otimes = \text{min} \times +$ we have

$$(0, \ 0) \otimes ((0, \ 0) \oplus (1, \ 1)) = (0, \ 0) \otimes (1, \ 1) = (0, \ 1)$$

but

$$((0, \ 0) \otimes (0, \ 0)) \oplus ((0, \ 0) \otimes (1, \ 1)) = (0, \ 0) \oplus (0, \ 1) = (0, \ 0)$$

Another construction

Suppose that $(S, \oplus_S)$ and $(T, \oplus_T)$ are both commutative and idempotent semigroups. Recall that $S \uplus T$ represents the disjoint union of sets S and T. That is,

$$S \uplus T \equiv \{\mathrm{inl}(s) \mid s \in S\} \cup \{\mathrm{inr}(t) \mid t \in T\}.$$

Define the operation $\oplus \equiv \oplus_S + \oplus_T$ over $S \uplus T$ as

$$\begin{array}{rcl} \mathrm{inl}(s) \oplus \mathrm{inl}(s') & \equiv & \mathrm{inl}(s \oplus_S s') \\ \mathrm{inr}(t) \oplus \mathrm{inr}(t') & \equiv & \mathrm{inr}(t \oplus_T t') \\ \mathrm{inl}(s) \oplus \mathrm{inr}(t) & \equiv & \mathrm{inl}(s) \\ \mathrm{inr}(t) \oplus \mathrm{inl}(s) & \equiv & \mathrm{inl}(s) \end{array}$$

Homework 1. Due 31 October.

1. Show that $\leqslant^L_\oplus$ is a partial order, where this is defined in the usual way as $x \leqslant^L_\oplus y \equiv x = x \oplus y$.
2. When is $\leqslant^L_\oplus$ is a total order?
3. Does $\leqslant^L_\oplus$ have a least element? A greatest element?
4. Suppose now that we have two semirings, $(S,\ \oplus_S,\ \otimes_S)$ and $(T,\ \oplus_T,\ \otimes_T)$. We want to define a combinator that will produce a semiring

   $$(S \uplus T,\ \oplus,\ \otimes).$$

   How would you define $\otimes$ from $\otimes_S$ and $\otimes_T$?
5. Can you give an informal interpretation for the resulting semiring?