L11: Algebraic Path Problems with applications to Internet Routing
Lecture 6
CAS Part II

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Distributivity?

Theorem: If $\oplus_S$ is commutative and selective, then

$$LD((S, \oplus_S, \otimes_S) \not\succ (T, \oplus_T, \otimes_T)) \iff$$

$$LD(S, \oplus_S, \otimes_S) \land LD(T, \oplus_T, \otimes_T) \land (LC(S, \otimes_S) \lor RK(T, \otimes_T))$$

$$RD((S, \oplus_S, \otimes_S) \not\prec (T, \oplus_T, \otimes_T)) \iff$$

$$RD(S, \oplus_S, \otimes_S) \land RD(T, \oplus_T, \otimes_T) \land (RC(S, \otimes_S) \lor RK(T, \otimes_T))$$

Left and Right Cancellative

$$LC(X, \bullet) \equiv \forall a, b, c \in X, c \bullet a = c \bullet b \Rightarrow a = b$$

$$RC(X, \bullet) \equiv \forall a, b, c \in X, a \bullet c = b \bullet c \Rightarrow a = b$$

Left and Right Constant

$$LK(X, \bullet) \equiv \forall a, b, c \in X, c \bullet a = c \bullet b$$

$$RK(X, \bullet) \equiv \forall a, b, c \in X, a \bullet c = b \bullet c$$
Why bisemigroups?

But wait! How could any semiring satisfy either of these properties?

\[
\begin{align*}
\text{LC}(X, \cdot) & \equiv \forall a, b, c \in X, c \cdot a = c \cdot b \Rightarrow a = b \\
\text{LK}(X, \cdot) & \equiv \forall a, b, c \in X, c \cdot a = c \cdot b
\end{align*}
\]

- For \(\text{LC}\), note that we always have \(\overline{0} \otimes a = \overline{0} \otimes b\), so \(\text{LC}\) could only hold when \(S = \{\overline{0}\}\).
- For \(\text{LK}\), let \(a = \overline{1}\) and \(b = \overline{0}\) and \(\text{LK}\) leads to the conclusion that every \(c\) is equal to \(\overline{0}\) (again!).

Normally we will add a zero and/or a one as the last step(s) of constructing a semiring. Alternatively, we might want to complicate our properties so that things work for semirings. A design trade-off!
Proof of $\iff$ for $\mathsf{LD}$ (Very carefully ...)

Assume

1. $\mathsf{LD}(S, \oplus_S, \otimes_S)$
2. $\mathsf{LD}(T, \oplus_T, \otimes_T)$
3. $\mathsf{LC}(S, \otimes_S) \lor \mathsf{LK}(T, \otimes_T)$
4. $\mathsf{IP}(S, \oplus_S)$.

Let $\oplus \equiv \oplus_S \rightarrow \oplus_T$ and $\otimes \equiv \otimes_S \times \otimes_T$. Suppose

$$(s_1, t_1), (s_2, t_2), (s_3, t_3) \in S \times T.$$ 

We want to show that

$$\text{lhs} \equiv (s_1, t_1) \otimes ((s_2, t_2) \oplus (s_3, t_3)) = ((s_1, t_1) \otimes (s_2, t_2)) \oplus ((s_1, t_1) \otimes (s_3, t_3)) \equiv \text{rhs}$$
Proof of $\leftrightarrow$ for $\mathbb{LD}$

We have

$$
\text{lhs} \quad \equiv \quad (s_1, t_1) \otimes ((s_2, t_2) \oplus (s_3, t_3)) \\
= \quad (s_1, t_1) \otimes (s_2 \oplus_{S} s_3, t_{\text{lhs}}) \\
= \quad (s_1 \otimes_{S} (s_2 \oplus_{S} s_3), t_1 \otimes_{T} t_{\text{lhs}})
$$

$$
\text{rhs} \quad \equiv \quad ((s_1, t_1) \otimes (s_2, t_2)) \oplus ((s_1, t_1) \otimes (s_3, t_3)) \\
= \quad (s_1 \otimes_{S} s_2, t_1 \otimes_{T} t_2) \oplus (s_1 \otimes_{S} s_3, t_1 \otimes_{T} t_3) \\
= \quad ((s_1 \otimes_{S} s_2) \oplus_{S} (s_1 \otimes_{S} s_3), t_{\text{rhs}}) \\
\equiv (1) \quad (s_1 \otimes_{S} (s_2 \oplus_{S} s_3), t_{\text{rhs}})
$$

where $t_{\text{lhs}}$ and $t_{\text{rhs}}$ are determined by the appropriate case in the definition of $\oplus$. Finally, note that

$$
\text{lhs} = \text{rhs} \iff t_{\text{rhs}} = t_1 \otimes t_{\text{lhs}}.
$$
Proof by cases on $s_2 \oplus_T s_3$

Case 1: $s_2 = s_2 \oplus_T s_3 = s_3$. Then $t_{lhs} = t_2 \oplus_T t_3$ and

$$t_1 \otimes_T t_{lhs} = t_1 \otimes_T (t_2 \oplus_T t_3) = (2) (t_1 \otimes_T t_2) \oplus_T (t_1 \otimes_T t_3).$$

Since $s_2 = s_3$ we have $s_1 \otimes_T s_2 = s_1 \otimes_T s_3$ and

$$s_1 \otimes_T s_2 = (4) (s_1 \otimes_T s_2) \oplus_T (s_1 \otimes_T s_3) = (4) s_1 \otimes_T s_3.$$

Therefore,

$$t_{rhs} = (t_1 \otimes_T t_2) \oplus (t_1 \otimes_T t_3) = t_1 \otimes_T t_{lhs}.$$

Case 2: $s_2 = s_2 \oplus_T s_3 \neq s_3$. Then $t_{lhs} = t_2$ and

$$t_1 \otimes_T t_{lhs} = t_1 \otimes_T t_2.$$

Since $s_2 = s_2 \oplus_T s_3$ we have

$$s_1 \otimes_T s_2 = s_1 \otimes_T (s_2 \oplus_T s_3) = (1) (s_1 \otimes_T s_2) \oplus_T (s_1 \otimes_T s_3).$$
Case 2.1 $s_1 \otimes_S s_2 \neq s_1 \otimes_S s_3$. Then $t_{\text{rhs}} = t_1 \otimes_T t_2 = t_1 \otimes_T t_{\text{lhs}}$.

Case 2.2 $s_1 \otimes_S s_2 = s_1 \otimes_S s_3$. Then

$$t_{\text{rhs}} = (t_1 \otimes_T t_2) \oplus_T (t_1 \otimes_T t_3) = (2) \quad t_1 \otimes_T (t_2 \oplus_T t_3)$$

We need to consider two subcases.

Case 2.2.1: Assume $\mathbb{L}\mathbb{C}(S, \otimes_S)$. But $s_1 \otimes_S s_2 = s_1 \otimes_S s_3 \Rightarrow s_2 = s_3$, which is a contradiction.

Case 2.2.2: Assume $\mathbb{L}\mathbb{K}(T, \otimes_T)$. In this case we know

$$\forall a, b \in X, \quad t_1 \otimes_T a = t_1 \otimes_T b.$$ 

Letting $a = t_2 \oplus_T t_3$ and $b = t_2$ we have

$$t_{\text{rhs}} = t_1 \otimes_T (t_2 \oplus_T t_3) = t_1 \otimes_T t_2 = t_1 \otimes_T t_{\text{lhs}}.$$ 

Case 3: $s_2 \neq s_2 \oplus_S s_3 = s_3$. Similar to Case 2.
Other direction, $\Rightarrow$ (Very carefully ...)

Prove this:

$$\neg \mathsf{LD}(S, \oplus_S, \otimes_S) \lor \neg \mathsf{LD}(T, \oplus_T, \otimes_T) \lor (\neg \mathsf{LC}(S, \otimes_S) \land \neg \mathsf{LK}(T, \otimes_T))$$

$$\Rightarrow \neg \mathsf{LD}((S, \oplus_S, \otimes_S) \not\cong (T, \oplus_T, \otimes_T)).$$

Case 1: $\neg \mathsf{LD}(S, \oplus_S, \otimes_S)$. That is

$$\exists a, b, c \in S, \ a \otimes_S (b \oplus_S c) \neq (a \otimes_S b) \oplus_S (a \otimes_S c).$$

Pick any $t \in T$. Then for some $t_1, t_2, t_3 \in T$ we have

$$\begin{align*}
(a, t) \otimes ((b, t) \oplus (c, t)) \\
= (a, t) \otimes (b \oplus_S c, t_1) \\
= (a, \otimes_S (b \oplus_S c), t_2) \\
\neq ((a \otimes_S b) \oplus_S (a \otimes_S c), t_3) \\
= (a \otimes_S b, t \otimes_T t) \oplus (a \otimes_S c, t \otimes_T t) \\
= ((a, t) \otimes (b, t)) \oplus ((a, t) \otimes (c, t))
\end{align*}$$

Case 2: $\neg \mathsf{LD}(T, \oplus_T, \otimes_T)$. Similar.
Case 3: \((-\text{LC}(S, \otimes_S) \land -\text{LK}(T, \otimes_T)) \).
That is

\[ \exists a, b, c \in S, \ c \otimes_S a = c \otimes_S b \land a \neq b \]

and

\[ \exists x, y, z \in T, \ z \otimes_T x \neq z \otimes_T y. \]

Since \( \oplus_S \) is selective and \( a \neq b \), we have \( a = a \oplus_S b \) or \( b = a \oplus_S b \).
Case 3.1: Assume \( a = a \oplus_S b \neq b \).
Suppose that \( t_1, t_2, t_3 \in T \). Then

\[
\text{lhs} \equiv (c, t_1) \otimes ((a, t_2) \oplus (b, t_3))
= (c, t_1) \otimes (a, t_2)
= (c \otimes_S a, t_1 \otimes_T t_2)
\]

\[
\text{rhs} \equiv ((c, t_1) \otimes (a, t_2)) \oplus ((c, t_1) \otimes (b, t_3))
= (c \otimes_S a, t_1 \otimes_T t_2) \oplus (c \otimes_S b, t_1 \otimes_T t_3)
= (c \otimes_S a, (t_1 \otimes_T t_2) \oplus_T (t_1 \otimes_T t_3))
\]
Our job now is to select \( t_1, t_2, t_3 \) so that

\[
t_{\text{lhs}} \equiv t_1 \otimes_T t_2 \neq (t_1 \otimes_T t_2) \oplus_T (t_1 \otimes_T t_3) \equiv t_{\text{rhs}}.
\]

We don’t have very much to work with! Only

\[
\exists x, y, z \in T, \ z \otimes_T x \neq z \otimes_T y.
\]

In addition, we can assume \( \mathbb{LD}(T, \oplus_T, \otimes_T) \) (otherwise, use Case 2!), so

\[
t_{\text{rhs}} = t_1 \otimes_T (t_2 \oplus_T t_3).
\]
We need to select $t_1$, $t_2$, $t_3$ so that

$$t_{\text{lhs}} = t_1 \otimes_T t_2 \neq t_1 \otimes_T (t_2 \oplus_T t_3) = t_{\text{rhs}}.$$ 

Case 3.1.1: $z \otimes_T x = z \otimes_T (x \oplus_T y)$. Then letting $t_1 = z$, $t_2 = y$, and $t_3 = x$ we have

$$t_{\text{lhs}} = z \otimes_T y \neq z \otimes_T x = z \otimes_T (x \oplus_T y) = z \otimes_T (y \oplus_T x) = t_{\text{rhs}}.$$ 

Case 3.1.2: $z \otimes_T x \neq z \otimes_T (x \oplus_T y)$. Then letting $t_1 = z$, $t_2 = x$, and $t_3 = y$ we have

$$t_{\text{lhs}} = z \otimes_T x \neq z \otimes_T (x \oplus_T y) = t_{\text{rhs}}.$$
Case 3.2: Assume $b = a \oplus_S b \neq a$.

Suppose that $t_1, t_2, t_3 \in T$. Then

\[
\text{lhs} \equiv (c, t_1) \otimes ((a, t_2) \oplus (b, t_3)) \\
= (c, t_1) \otimes (b, t_3) \\
= (c \otimes_S b, t_1 \otimes_T t_3)
\]

\[
\text{rhs} \equiv ((c, t_1) \otimes (a, t_2)) \oplus ((c, t_1) \otimes (b, t_3)) \\
= (c \otimes_S a, t_1 \otimes_T t_2) \oplus (c \otimes_S b, t_1 \otimes_T t_3) \\
= (c \otimes_S b, (t_1 \otimes_T t_2) \oplus_T (t_1 \otimes_T t_3)) \\
= (c \otimes_S b, t_1 \otimes_T (t_2 \oplus_T t_3))
\]
We need to select \( t_1, t_2, t_3 \) so that

\[
t_{\text{lhs}} = t_1 \otimes_T t_3 \neq t_1 \otimes_T (t_2 \oplus_T t_3) = t_{\text{rhs}}.
\]

Case 3.2.1: \( z \otimes_T x = z \otimes_T (x \oplus_T y) \). Then letting \( t_1 = z, t_2 = x, \) and \( t_3 = y \) we have

\[
t_{\text{lhs}} = z \otimes_T y \neq z \otimes_T x = z \otimes_T (x \oplus_T y) = t_{\text{rhs}}.
\]

Case 3.2.2: \( z \otimes_T x \neq z \otimes_T (x \oplus_T y) \). Letting \( t_1 = z, t_2 = y, \) and \( t_3 = x \) we have

\[
t_{\text{lhs}} = z \otimes_T x \neq z \otimes_T (x \oplus_T y) = z \otimes_T (y \oplus_T x) = t_{\text{rhs}}.
\]
Computing Counter Examples

Note that from \((a, b, c)\) such \(c \otimes_S a = c \otimes_S b \land a \neq b\) and \((x, y, z)\) such that \(z \otimes_T x \neq z \otimes_T y\) our proof computes a counter example to LD as

\[
\text{if } a = a \oplus_S b \\
\text{then if } z \otimes_T x = (z \otimes_T x) \oplus_T (z \otimes_T y) \\
\quad \text{then } ((a, z), (b, y), (c, x)) \\
\quad \text{else } ((a, z), (b, x), (c, y)) \\
\text{else if } z \otimes_T x = (z \otimes_T x) \oplus_T (z \otimes_T y) \\
\quad \text{then } ((a, z), (b, x), (c, y)) \\
\quad \text{else } ((a, z), (b, y), (c, x))
\]
Examples

True or counter example

<table>
<thead>
<tr>
<th>name</th>
<th>$S$</th>
<th>$\oplus$</th>
<th>$\otimes$</th>
<th>$\text{LD}$</th>
<th>$\text{LC}(S, \otimes)$</th>
<th>$\text{LK}(S, \otimes)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>min_plus</td>
<td>$\mathbb{N}$</td>
<td>min</td>
<td>+</td>
<td>$\ast$</td>
<td>$\ast$</td>
<td>$(0, 0, 1)$</td>
</tr>
<tr>
<td>max_min</td>
<td>$\mathbb{N}$</td>
<td>max</td>
<td>min</td>
<td>$\ast$</td>
<td>$(0, 0, 1)$</td>
<td>$(1, 0, 1)$</td>
</tr>
</tbody>
</table>

For example, $(0, 0, 1)$ is a counter example for $\text{LC}(\mathbb{N}, \min)$ since $0 \min 0 = 0 \min 1$, but $0 \neq 1$.

Let’s turn the crank

$$\text{LD}((\text{min}_\text{plus} \mapsto \text{max}_\text{min}))$$

$\iff$

$$\text{LD}((\text{min}_\text{plus}) \land \text{LD}((\text{max}_\text{min}) \land (\text{LC}(\mathbb{N}, +) \lor \text{LK}(\mathbb{N}, \min))))$$

$\iff$ TRUE
Examples

Another turn of the crank

\[ \text{LD}(\text{max}_\min \times \text{min}_\plus) \]
\[ \iff \text{LD}(\text{max}_\min) \land \text{LD}(\text{min}_\plus) \land (\text{LC}(N, \text{min}) \lor \text{LK}(N, +)) \]
\[ \iff \text{FALSE} \]

Note that the counter examples to \text{LC} and \text{LK} can be plugged into the proof above to produce the a counter example to \text{LD},

\[ ((0, 0), (0, 0), (1, 1)) \]

and sure enough, with \( \oplus = \text{max} \times \text{min} \) and \( \otimes = \text{min} \times + \) we have

\[ (0, 0) \otimes ((0, 0) \oplus (1, 1)) = (0, 0) \otimes (1, 1) = (0, 1) \]

but

\[ ((0, 0) \otimes (0, 0)) \oplus ((0, 0) \otimes (1, 1)) = (0, 0) \oplus (0, 1) = (0, 0) \]
Another construction

Suppose that \((S, \oplus_S)\) and \((T, \oplus_T)\) are both commutative and idempotent semigroups. Recall that \(S \uplus T\) represents the disjoint union of sets \(S\) and \(T\). That is,

\[
S \uplus T \equiv \{\text{inl}(s) \mid s \in S\} \cup \{\text{inr}(t) \mid t \in T\}.
\]

Define the operation \(\oplus \equiv \oplus_S + \oplus_T\) over \(S \uplus T\) as

\[
\begin{align*}
\text{inl}(s) \oplus \text{inl}(s') & \equiv \text{inl}(s \oplus_S s') \\
\text{inr}(t) \oplus \text{inr}(t') & \equiv \text{inr}(t \oplus_T t') \\
\text{inl}(s) \oplus \text{inr}(t) & \equiv \text{inl}(s) \\
\text{inr}(t) \oplus \text{inl}(s) & \equiv \text{inl}(s)
\end{align*}
\]
1. Show that $\leq_{L+}$ is a partial order, where this is defined in the usual way as $x \leq_{L+} y \equiv x = x \oplus y$.

2. When is $\leq_{L+}$ a total order?

3. Does $\leq_{L+}$ have a least element? A greatest element?

4. Suppose now that we have two semirings, $(S, \oplus_S, \otimes_S)$ and $(T, \oplus_T, \otimes_T)$. We want to define a combinator that will produce a semiring

   $$(S \uplus T, \oplus, \otimes).$$

   How would you define $\otimes$ from $\otimes_S$ and $\otimes_T$?

5. Can you give an informal interpretation for the resulting semiring?