L11: Algebraic Path Problems with applications to Internet Routing Lecture 5 Introduction to Combinators for Algebraic Structures (CAS)

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Michaelmas Term, 2018

Semigroup properties (so far). Call this set of properties $\mathbb{P}_0^{SG}$

$$
\begin{array}{rcl}
\mathbb{AS}(S,\ \bullet) & \equiv & \forall a, b, c \in S,\ a \bullet (b \bullet c) = (a \bullet b) \bullet c \\
\mathbb{IID}(S,\ \bullet,\ \alpha) & \equiv & \forall a \in S,\ a = \alpha \bullet a = a \bullet \alpha \\
\mathbb{ID}(S,\ \bullet) & \equiv & \exists \alpha \in S,\ \mathbb{IID}(S,\ \bullet,\ \alpha) \\
\mathbb{IAN}(S,\ \bullet,\ \omega) & \equiv & \forall a \in S,\ \omega = \omega \bullet a = a \bullet \omega \\
\mathbb{AN}(S,\ \bullet) & \equiv & \exists \omega \in S,\ \mathbb{IAN}(S,\ \bullet,\ \omega) \\
\mathbb{CM}(S,\ \bullet) & \equiv & \forall a, b \in S,\ a \bullet b = b \bullet a \\
\mathbb{SL}(S,\ \bullet) & \equiv & \forall a, b \in S,\ a \bullet b \in \{a,\ b\} \\
\mathbb{IP}(S,\ \bullet) & \equiv & \forall a \in S,\ a \bullet a = a
\end{array}
$$

Bisemigroup properties (so far). Call this set of properties $\mathbb{P}_0^{BS}$

$$
\begin{array}{rcl}
\mathbb{LD}(S,\ \oplus,\ \otimes) & \equiv & \forall a, b, c \in S,\ a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c) \\
\mathbb{RD}(S,\ \oplus,\ \otimes) & \equiv & \forall a, b, c \in S,\ (a \oplus b) \otimes c = (a \otimes c) \oplus (b \otimes c) \\
\mathbb{ZA}(S,\ \oplus,\ \otimes) & \equiv & \exists \overline{0} \in S,\ \mathbb{IID}(S,\ \oplus,\ \overline{0}) \wedge \mathbb{IAN}(S,\ \otimes,\ \overline{0}) \\
\mathbb{OA}(S,\ \oplus,\ \otimes) & \equiv & \exists \overline{1} \in S,\ \mathbb{IID}(S,\ \otimes,\ \overline{1}) \wedge \mathbb{IAN}(S,\ \oplus,\ \overline{1})
\end{array}
$$

Start with an (expandable) set of base structures

Semigroups

S	$\bullet$	α	ω	$\mathbb{CM}$	$\mathbb{SL}$	$\mathbb{IP}$
S	left				$\star$	$\star$
S	right				$\star$	$\star$
$\mathbb{N}$	min		0	$\star$	$\star$	$\star$
$\mathbb{N}$	max	0		$\star$	$\star$	$\star$
$\mathbb{N}$	$+$	0		$\star$		

Bisemigroups

S	$\oplus$	$\otimes$	$\overline{0}$	$\overline{1}$	$\mathbb{LD}$	$\mathbb{RD}$	$\mathbb{ZA}$	$\mathbb{OA}$
$\mathbb{N}$	min	$+$		0	$\star$	$\star$		$\star$
$\mathbb{N}$	max	$+$	0	0	$\star$	$\star$		
$\mathbb{N}$	max	min	0		$\star$	$\star$	$\star$	
$\mathbb{N}$	min	max		0	$\star$	$\star$		$\star$

CAS idea

- We want to develop a set of combinators for constructing new semigroups and bisemigroups.
- A CAS expression will be built from base structures and combinators.
- We want the collection of combinators to be **closed** in the following sense:
  - For each property $\mathbb{Q}$ we can **compute** if a CAS expression satisfies $\mathbb{Q}$ or if it satisfies $\neg\mathbb{Q}$.

Add identity

$$\mathrm{AddId}(\alpha,\ (S,\ \bullet)) \equiv (S \uplus \{\alpha\}, \bullet_{\alpha}^{\mathrm{id}})$$

where $A \uplus B \equiv \{\mathrm{inl}(a) \mid a \in A\} \cup \{\mathrm{inr}(b) \mid b \in B\}$ and

$$a \bullet_{\alpha}^{\mathrm{id}} b \quad \equiv \quad \begin{cases} a & (\text{if } b = \mathrm{inr}(\alpha)) \\ b & (\text{if } a = \mathrm{inr}(\alpha)) \\ \mathrm{inl}(x \bullet y) & (\text{if } a = \mathrm{inl}(x),\ b = \mathrm{inl}(y)) \end{cases}$$

Easy Exercises

$$\begin{array}{rcl} \mathbb{AS}(\mathrm{AddId}(\alpha,\ (S,\ \bullet))) & \Leftrightarrow & \mathbb{AS}(S, \bullet) \\ \mathbb{ID}(\mathrm{AddId}(\alpha,\ (S,\ \bullet))) & \Leftrightarrow & \mathbb{TRUE} \\ \mathbb{AN}(\mathrm{AddId}(\alpha,\ (S,\ \bullet))) & \Leftrightarrow & \mathbb{AN}(S, \bullet) \\ \mathbb{CM}(\mathrm{AddId}(\alpha,\ (S,\ \bullet))) & \Leftrightarrow & \mathbb{CM}(S, \bullet) \\ \mathbb{IP}(\mathrm{AddId}(\alpha,\ (S,\ \bullet))) & \Leftrightarrow & \mathbb{IP}(S, \bullet) \\ \mathbb{SL}(\mathrm{AddId}(\alpha,\ (S,\ \bullet))) & \Leftrightarrow & \mathbb{SL}(S, \bullet) \end{array}$$

Adding an annihilator

$$\mathrm{AddAn}(\omega,\ (S,\ \bullet)) \equiv (S \uplus \{\omega\}, \bullet^{\mathrm{an}}_{\omega})$$

where

$$a \bullet^{\mathrm{an}}_{\omega} b \quad \equiv \quad \begin{cases} \mathrm{inr}(\omega) & (\text{if } b = \mathrm{inr}(\omega)) \\ \mathrm{inr}(\omega) & (\text{if } a = \mathrm{inr}(\omega)) \\ \mathrm{inl}(x \bullet y) & (\text{if } a = \mathrm{inl}(x),\ b = \mathrm{inl}(y)) \end{cases}$$

Easy Exercises

$$\begin{array}{rcl} \mathbb{AS}(\mathrm{AddAn}(\alpha,\ (S,\ \bullet))) & \Leftrightarrow & \mathbb{AS}(S, \bullet) \\ \mathbb{ID}(\mathrm{AddAn}(\alpha,\ (S,\ \bullet))) & \Leftrightarrow & \mathbb{ID}(S, \bullet) \\ \mathbb{AN}(\mathrm{AddAn}(\alpha,\ (S,\ \bullet))) & \Leftrightarrow & \mathbb{TRUE} \\ \mathbb{CM}(\mathrm{AddAn}(\alpha,\ (S,\ \bullet))) & \Leftrightarrow & \mathbb{CM}(S, \bullet) \\ \mathbb{IP}(\mathrm{AddAn}(\alpha,\ (S,\ \bullet))) & \Leftrightarrow & \mathbb{IP}(S, \bullet) \\ \mathbb{SL}(\mathrm{AddAn}(\alpha,\ (S,\ \bullet))) & \Leftrightarrow & \mathbb{SL}(S, \bullet) \end{array}$$

Direct Product of Semigroups

Let $(S,\ \bullet)$ and $(T,\ \diamond)$ be semigroups.

Definition (Direct product semigroup)

The direct product is denoted

$$(S,\ \bullet) \times (T,\ \diamond) \equiv (S \times T,\ \star)$$

where

$$\star = \bullet \times \diamond$$

is defined as

$$(s_1,\ t_1) \star (s_2,\ t_2) = (s_1 \bullet s_2,\ t_1 \diamond t_2).$$

Easy exercises

$$\begin{array}{rcl}
\mathbb{AS}((S, \bullet) \times (T, \diamond)) & \Leftrightarrow & \mathbb{AS}(S, \bullet) \wedge \mathbb{AS}(T, \diamond) \\
\mathbb{ID}((S, \bullet) \times (T, \diamond)) & \Leftrightarrow & \mathbb{ID}(S, \bullet) \wedge \mathbb{ID}(T, \diamond) \\
\mathbb{AN}((S, \bullet) \times (T, \diamond)) & \Leftrightarrow & \mathbb{AN}(S, \bullet) \wedge \mathbb{AN}(T, \diamond) \\
\mathbb{CM}((S, \bullet) \times (T, \diamond)) & \Leftrightarrow & \mathbb{CM}(S, \bullet) \wedge \mathbb{CM}(T, \diamond) \\
\mathbb{IP}((S, \bullet) \times (T, \diamond)) & \Leftrightarrow & \mathbb{IP}(S, \bullet) \wedge \mathbb{IP}(T, \diamond)
\end{array}$$

What about $\mathbb{SL}$?

Consider the product of two selective semigroups, such as $(\mathbb{N}, \min) \times (\mathbb{N}, \max)$.

$$(10, 10) \star (1, 3) = (1, 10) \notin \{(10, 10), (1, 3)\}$$

The result in this case is not selective!

Direct product and $\mathbb{SL}$?

$$\mathbb{SL}((S, \bullet) \times (T, \diamond)) \iff (\mathbb{IR}(S, \bullet) \wedge \mathbb{IR}(T, \diamond)) \vee (\mathbb{IL}(S, \bullet) \wedge \mathbb{IL}(T, \diamond))$$

$$\begin{array}{lcl} \mathbb{IR} \quad \text{is right} & \equiv & \forall s, t \in S, s \bullet t = t \\ \mathbb{IL} \quad \text{is left} & \equiv & \forall s, t \in S, s \bullet t = s \end{array}$$

$$\begin{array}{lcl} \mathbb{IR}((S, \bullet) \times (T, \diamond)) & \iff & \mathbb{IR}(S, \bullet) \wedge \mathbb{IR}(T, \diamond) \\ \mathbb{IL}((S, \bullet) \times (T, \diamond)) & \iff & \mathbb{IL}(S, \bullet) \wedge \mathbb{IL}(T, \diamond) \end{array}$$

Remember : we have an implicit assumtion that $2 \leqslant |S|$.

Revisit other semigroup constructions ...

To **close** our simple collection $\{\mathrm{AddId},\ \mathrm{AddAn}\}$ of semigroup combinators we need

$$\mathbb{P}_1^{SG} \equiv \mathbb{P}_0^{SG} \cup \{\mathbb{IR},\ \mathbb{IL}\}$$

and

$$\begin{array}{rcl} \mathbb{IR}(\mathrm{AddId}(\alpha,\ (S,\ \bullet))) & \Leftrightarrow & \mathbb{FALSE} \\ \mathbb{IL}(\mathrm{AddId}(\alpha,\ (S,\ \bullet))) & \Leftrightarrow & \mathbb{FALSE} \\ \\ \mathbb{IR}(\mathrm{AddAn}(\alpha,\ (S,\ \bullet))) & \Leftrightarrow & \mathbb{FALSE} \\ \mathbb{IL}(\mathrm{AddAn}(\alpha,\ (S,\ \bullet))) & \Leftrightarrow & \mathbb{FALSE} \end{array}$$

Operations for adding a zero, a one

$$\text{AddZero}(\overline{0},\ (S,\ \oplus,\ \otimes)) \equiv (S \uplus \{\overline{0}\},\ \oplus_{\overline{0}}^{\text{id}},\ \otimes_{\overline{0}}^{\text{an}})$$

$$\text{AddOne}(\overline{1},\ (S,\ \oplus,\ \otimes)) \equiv (S \uplus \{\overline{1}\},\ \oplus_{\overline{1}}^{\text{an}},\ \otimes_{\overline{1}}^{\text{id}})$$

Easy Exercises

$$\begin{array}{lcl}
\mathbb{LD}(\text{AddZero}(\overline{0},\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & \mathbb{LD}(S,\ \oplus,\ \otimes) \\
\mathbb{RD}(\text{AddZero}(\overline{0},\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & \mathbb{RD}(S,\ \oplus,\ \otimes) \\
\mathbb{ZA}(\text{AddZero}(\overline{0},\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & \mathbb{TRUE} \\
\mathbb{OA}(\text{AddZero}(\overline{0},\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & \mathbb{OA}(S,\ \oplus,\ \otimes)
\end{array}$$

Easy Exercises?

Consider left distributivity ($\mathbb{LD}$)

a	b	c	$a \otimes_{\overline{0}}^{\text{an}} (b \oplus_{\overline{0}}^{\text{id}} c)$	$(a \otimes_{\overline{0}}^{\text{an}} b) \oplus_{\overline{0}}^{\text{id}} (a \otimes_{\overline{0}}^{\text{an}} c)$
$\text{inl}(a')$	$\text{inl}(b')$	$\text{inl}(c')$	$\text{inl}(a' \otimes (b' \oplus c'))$	$\text{inl}((a' \otimes b') \oplus (a' \otimes c'))$
$\text{inr}(\overline{0})$	$\text{inl}(b')$	$\text{inl}(c')$	$\text{inr}(\overline{0})$	$\text{inr}(\overline{0})$
$\text{inl}(a')$	$\text{inr}(\overline{0})$	$\text{inl}(c')$	$\text{inl}(a' \oplus c')$	$\text{inl}(a' \oplus c')$
$\text{inl}(a')$	$\text{inl}(b')$	$\text{inr}(\overline{0})$	$\text{inl}(a' \oplus b')$	$\text{inl}(a' \oplus b')$
$\text{inl}(a')$	$\text{inr}(\overline{0})$	$\text{inr}(\overline{0})$	$\text{inr}(\overline{0})$	$\text{inr}(\overline{0})$
$\text{inr}(\overline{0})$	$\text{inr}(\overline{0})$	$\text{inr}(\overline{0})$	$\text{inr}(\overline{0})$	$\text{inr}(\overline{0})$

However, adding a one is more complicated!

Consider left distributivity ($\mathbb{LD}$)

a	b	c	$a \otimes^{\mathrm{id}}_{\bar{1}} (b \oplus^{\mathrm{an}}_{\bar{1}} c)$	$(a \otimes^{\mathrm{id}}_{\bar{1}} b) \oplus^{\mathrm{an}}_{\bar{1}} (a \otimes^{\mathrm{id}}_{\bar{1}} c)$
$\mathrm{inl}(a')$	$\mathrm{inl}(b')$	$\mathrm{inl}(c')$	$\mathrm{inl}(a' \otimes (b' \oplus c'))$	$\mathrm{inl}((a' \otimes b') \oplus (a' \otimes c'))$
$\mathrm{inr}(\bar{1})$	$\mathrm{inl}(b')$	$\mathrm{inl}(c')$	$\mathrm{inl}(b' \oplus c')$	$\mathrm{inl}(b' \oplus c')$
$\mathrm{inl}(a')$	$\mathrm{inr}(\bar{1})$	$\mathrm{inl}(c')$	$\mathrm{inl}(a')$	$\mathrm{inl}((a' \oplus (a' \otimes c'))$
$\mathrm{inl}(a')$	$\mathrm{inl}(b')$	$\mathrm{inr}(\bar{1})$	$\mathrm{inl}(a')$	$\mathrm{inl}((a' \otimes b') \oplus a')$
$\mathrm{inl}(a')$	$\mathrm{inr}(\bar{1})$	$\mathrm{inr}(\bar{1})$	$\mathrm{inl}(a')$	$\mathrm{inl}(a' \oplus a')$
$\mathrm{inr}(\bar{1})$	$\mathrm{inr}(\bar{1})$	$\mathrm{inr}(\bar{1})$	$\mathrm{inr}(\bar{1})$	$\mathrm{inr}(\bar{1})$

Absorption

what does $a = (a \otimes b) \oplus a$ represent?

Let $a \leqslant b \equiv a = a \oplus b$. Then $a = (a \otimes b) \oplus a$ is telling us something else, that

$$a \leqslant a \otimes b.$$

That is, that multiplication is inflationary or non-decreasing.

$\mathbb{AB}$sorption properties (name is from lattice theory)

$$\begin{array}{lll} \mathbb{RAB}(S, \oplus, \otimes) & \equiv & \forall a, b \in S,\ a = (a \otimes b) \oplus a = a \oplus (a \otimes b) \\ \mathbb{LAB}(S, \oplus, \otimes) & \equiv & \forall a, b \in S,\ a = (b \otimes a) \oplus a = a \oplus (b \otimes a) \end{array}$$

To **close** our simple collection $\{\mathrm{AddZero},\ \mathrm{AddOne}\}$ of bisemigroup combinators we need

$$\mathbb{P}_1^{BS} \equiv \mathbb{P}_0^{BS} \cup \{\mathbb{RAB},\ \mathbb{LAB}\}.$$

Rules for absorption for $\mathrm{AddZero}$? Consider $\mathbb{RAB}$

AddZero

a	b	$(a \otimes_{\overline{0}}^{\mathrm{an}} b) \oplus_{\overline{0}}^{\mathrm{id}} a$	$a \oplus_{\overline{0}}^{\mathrm{id}} (a \otimes_{\overline{0}}^{\mathrm{an}} b)$
$\mathrm{inl}(a')$	$\mathrm{inl}(b')$	$\mathrm{inl}((a' \otimes b') \oplus a)$	$\mathrm{inl}(a' \oplus (a' \otimes b'))$
$\mathrm{inr}(\overline{0})$	$\mathrm{inl}(b')$	$\mathrm{inr}(\overline{0})$	$\mathrm{inr}(\overline{0})$
$\mathrm{inl}(a')$	$\mathrm{inr}(\overline{0})$	$\mathrm{inl}(a')$	$\mathrm{inl}(a')$
$\mathrm{inr}(\overline{0})$	$\mathrm{inr}(\overline{0})$	$\mathrm{inr}(\overline{0})$	$\mathrm{inr}(\overline{0})$

$$\mathbb{RAB}(\mathrm{AddZero}(\overline{0},\ (S,\ \oplus,\ \otimes))) \ \Leftrightarrow\ \mathbb{RAB}(S,\ \oplus,\ \otimes)$$
$$\mathbb{LAB}(\mathrm{AddZero}(\overline{0},\ (S,\ \oplus,\ \otimes))) \ \Leftrightarrow\ \mathbb{LAB}(S,\ \oplus,\ \otimes)$$

Rules for absorption for AddOne? Consider $\mathbb{RAB}$

AddOne

a	b	$(a \otimes_1^{\mathrm{id}} b) \oplus_1^{\mathrm{an}} a$	$a \oplus_1^{\mathrm{an}} (a \otimes_1^{\mathrm{id}} b)$
$\mathrm{inl}(a')$	$\mathrm{inl}(b')$	$\mathrm{inl}((a' \otimes b') \oplus a)$	$\mathrm{inl}(a' \oplus (a' \otimes b'))$
$\mathrm{inr}(\overline{1})$	$\mathrm{inl}(b')$	$\mathrm{inr}(\overline{1})$	$\mathrm{inr}(\overline{1})$
$\mathrm{inl}(a')$	$\mathrm{inr}(\overline{1})$	$\mathrm{inl}(a')$	$\mathrm{inl}(a' \oplus a')$
$\mathrm{inr}(\overline{1})$	$\mathrm{inr}(\overline{1})$	$\mathrm{inr}(\overline{1})$	$\mathrm{inr}(\overline{1})$

Property management for AddOne

$$
\begin{array}{rcl}
\mathbb{LD}(\mathrm{AddOne}(\overline{1},\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & \mathbb{LD}(S,\ \oplus,\ \otimes) \wedge \mathbb{RAB}(S,\ \oplus,\ \otimes) \\
 & & \wedge\ \mathbb{IP}(S,\ \oplus) \\
\mathbb{RD}(\mathrm{AddOne}(\overline{1},\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & \mathbb{RD}(S,\ \oplus,\ \otimes) \wedge \mathbb{LAB}(S,\ \oplus,\ \otimes) \\
 & & \wedge\ \mathbb{IP}(S,\ \oplus) \\
\mathbb{ZA}(\mathrm{AddOne}(\overline{1},\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & \mathbb{ZA}(S,\ \oplus,\ \otimes) \\
\mathbb{OA}(\mathrm{AddOne}(\overline{1},\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & \mathbb{TRUE} \\
\mathbb{RAB}(\mathrm{AddOne}(\overline{1},\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & \mathbb{RAB}(S,\ \oplus,\ \otimes) \wedge \mathbb{IP}(S,\ \oplus) \\
\mathbb{LAB}(\mathrm{AddOne}(\overline{1},\ (S,\ \oplus,\ \otimes))) & \Leftrightarrow & \mathbb{LAB}(S,\ \oplus,\ \otimes) \wedge \mathbb{IP}(S,\ \oplus)
\end{array}
$$

Lexicographic Product of Semigroups

Lexicographic product semigroup

Suppose that semigroup $(S, \bullet)$ is commutative, idempotent, and selective and that $(T, \diamond)$ is a semigroup.

$$(S, \bullet) \ \vec{\times} \ (T, \diamond) \equiv (S \times T, \star)$$

where $\star \equiv \bullet \vec{\times} \diamond$ is defined as

$$(s_1, t_1) \star (s_2, t_2) = \begin{cases} (s_1 \bullet s_2, \ t_1 \diamond t_2) & s_1 = s_1 \bullet s_2 = s_2 \\ (s_1 \bullet s_2, \ t_1) & s_1 = s_1 \bullet s_2 \neq s_2 \\ (s_1 \bullet s_2, \ t_2) & s_1 \neq s_1 \bullet s_2 = s_2 \end{cases}$$

Examples

$(\mathbb{N}, \min) \ \vec{\times} \ (\mathbb{N}, \min)$

$$
\begin{aligned}
(1,\ 17) \star (2,3) &= (1,17) \\
(2,\ 17) \star (2,3) &= (2,3) \\
(2,\ 3) \star (2,3) &= (2,3)
\end{aligned}
$$

$(\mathbb{N}, \min) \ \vec{\times} \ (\mathbb{N}, \max)$

$$
\begin{aligned}
(1,\ 17) \star (2,3) &= (1,17) \\
(2,\ 17) \star (2,3) &= (2,17) \\
(2,\ 3) \star (2,3) &= (2,3)
\end{aligned}
$$

$(\mathbb{N}, \max) \ \vec{\times} \ (\mathbb{N}, \min)$

$$
\begin{aligned}
(1,\ 17) \star (2,3) &= (2,3) \\
(2,\ 17) \star (2,3) &= (2,3) \\
(2,\ 3) \star (2,3) &= (2,3)
\end{aligned}
$$

Assuming $\mathbb{CM}(S, \bullet) \wedge \mathbb{SL}(S, \bullet)$

$$\begin{array}{rcl}
\mathbb{AS}((S, \bullet) \vec{\times} (T, \diamond)) & \Leftrightarrow & \mathbb{AS}(S, \bullet) \wedge \mathbb{AS}(T, \diamond) \\
\mathbb{ID}((S, \bullet) \vec{\times} (T, \diamond)) & \Leftrightarrow & \mathbb{ID}(S, \bullet) \wedge \mathbb{ID}(T, \diamond) \\
\mathbb{AN}((S, \bullet) \vec{\times} (T, \diamond)) & \Leftrightarrow & \mathbb{AN}(S, \bullet) \wedge \mathbb{AN}(T, \diamond) \\
\mathbb{CM}((S, \bullet) \vec{\times} (T, \diamond)) & \Leftrightarrow & \mathbb{CM}(T, \diamond) \\
\mathbb{IP}((S, \bullet) \vec{\times} (T, \diamond)) & \Leftrightarrow & \mathbb{IP}(T, \diamond) \\
\mathbb{SL}((S, \bullet) \vec{\times} (T, \diamond)) & \Leftrightarrow & \mathbb{SL}(T, \diamond) \\
\mathbb{IR}((S, \bullet) \vec{\times} (T, \diamond)) & \Leftrightarrow & \mathbb{FALSE} \\
\mathbb{IL}((S, \bullet) \vec{\times} (T, \diamond)) & \Leftrightarrow & \mathbb{FALSE}
\end{array}$$

All easy, except for $\mathbb{AS}$ (very tedious!). We are assuming commutativity and selectivity in order to guarantee associativity.

Lexicographic product for Bi-Semigroups

Assume $\mathbb{AS}(S, \oplus_S) \wedge \mathbb{AS}(T, \oplus_T) \wedge \mathbb{CM}(S, \oplus_S) \wedge \mathbb{SL}(S, \oplus_S)$

Let

$$(S, \oplus_S, \otimes_S) \vec{\times} (T, \oplus_T, \otimes_T) \equiv (S \times T, \oplus_S \vec{\times} \oplus_T, \otimes_S \times \otimes_T)$$

That is, the additive component is a lexicographic product, and the multiplicative component is a direct product.

Examples

$\oplus = \min \vec{\times} \max,\ \otimes = + \times \min$

$$\begin{aligned}(3,10) \otimes ((17,21) \oplus (11,4)) &= (3,10) \otimes (11,4)\\ &= (14,4)\end{aligned}$$

$$\begin{aligned}((3,10) \otimes (17,21)) \oplus ((3,10) \otimes (11,4)) &= (20,10) \oplus (14,4)\\ &= (14,4)\end{aligned}$$

$\oplus = \max \vec{\times} \min,\ \otimes = \min \times +$

$$\begin{aligned}(3,10) \otimes ((17,21) \oplus (11,4)) &= (3,10) \otimes (17,21)\\ &= (3,31)\end{aligned}$$

$$\begin{aligned}((3,10) \otimes (17,21)) \oplus ((3,10) \otimes (11,4)) &= (3,31) \oplus (3,14)\\ &= (3,14)\end{aligned}$$

Distributivity?

Theorem: If $\oplus_S$ is commutative and selective, then

$$\mathbb{LD}((S,\ \oplus_S,\ \otimes_S) \vec{\times} (T,\ \oplus_T,\ \otimes_T)) \Leftrightarrow$$
$$\mathbb{LD}(S,\ \oplus_S,\ \otimes_S) \wedge \mathbb{LD}(T,\ \oplus_T,\ \otimes_T) \wedge (\mathbb{LC}(S,\ \otimes_S) \vee \mathbb{LK}(T,\ \otimes_T))$$

$$\mathbb{RD}((S,\ \oplus_S,\ \otimes_S) \vec{\times} (T,\ \oplus_T,\ \otimes_T)) \Leftrightarrow$$
$$\mathbb{RD}(S,\ \oplus_S,\ \otimes_S) \wedge \mathbb{RD}(T,\ \oplus_T,\ \otimes_T) \wedge (\mathbb{RC}(S,\ \otimes_S) \vee \mathbb{RK}(T,\ \otimes_T))$$

Left and Right Cancellative

$$\mathbb{LC}(X,\ \bullet) \equiv \forall a, b, c \in X,\ c \bullet a = c \bullet b \Rightarrow a = b$$
$$\mathbb{RC}(X,\ \bullet) \equiv \forall a, b, c \in X,\ a \bullet c = b \bullet c \Rightarrow a = b$$

Left and Right Constant

$$\mathbb{LK}(X,\ \bullet) \equiv \forall a, b, c \in X,\ c \bullet a = c \bullet b$$
$$\mathbb{RK}(X,\ \bullet) \equiv \forall a, b, c \in X,\ a \bullet c = b \bullet c$$