L11: Algebraic Path Problems with applications to Internet Routing Lecture 4

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Lecture 4 : Two interesting Semirings

- Martelli's semiring for computing minimal cut sets
- The mini-max semiring

Cut Sets

Let $G = (V, E)$ be a directed graph.

- A cut set $C \subseteq E$ for nodes i and j is a set of arcs such there is no path from i to j in the graph $(V, E - C)$.
- C is minimal if no proper subset of C is an arc cut set.

Martelli's Semiring

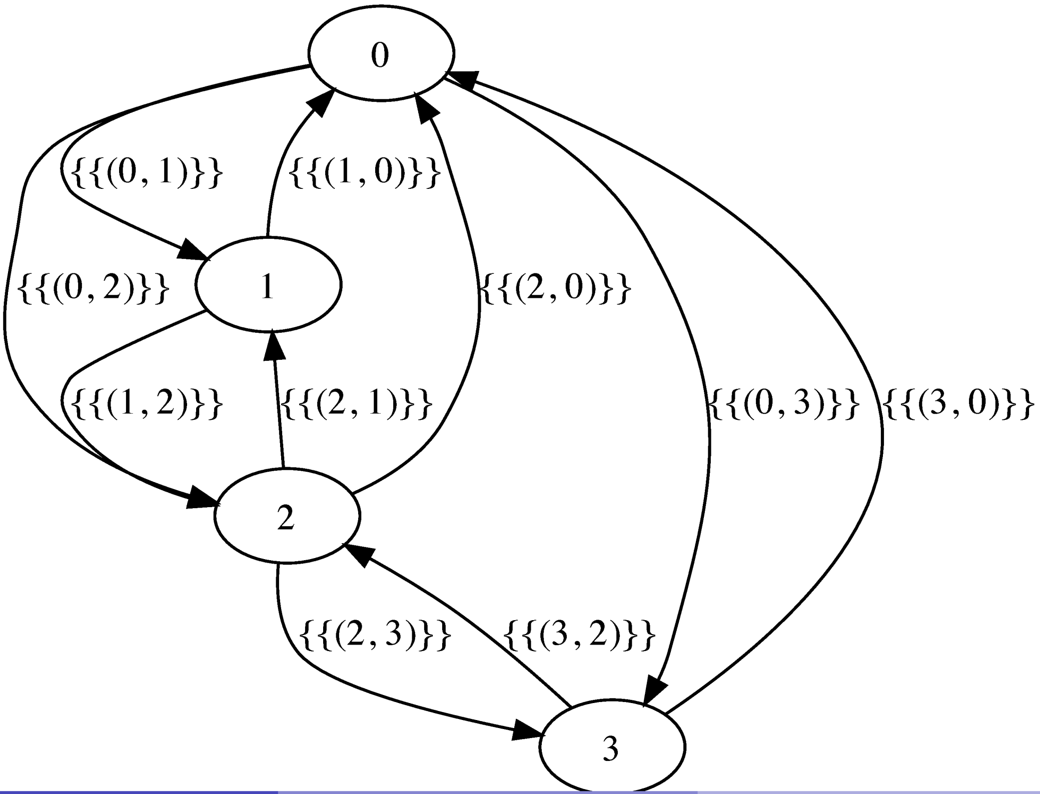
Let $G = (V, E)$ be a directed graph.

$$\begin{array}{rcl} \mathrm{M} & \equiv & (S, \oplus, \otimes, \overline{0}, \overline{1}) \\ S & \equiv & \{X \in 2^{2^E} \mid \forall U, V \in X, U \subseteq V \implies U = V\} \\ X \oplus Y & \equiv & \text{remove all supersets from } \{U \cup V \mid U \in X, V \in Y\} \\ X \otimes Y & \equiv & \text{remove all supersets from } X \cup Y \\ \overline{0} & \equiv & \{\{\}\} \\ \overline{1} & \equiv & \{\} \end{array}$$

What does it do?

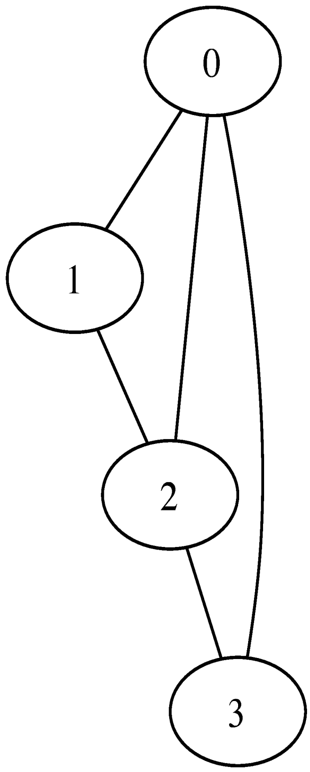
- If every arc (i, j) is has weight $\mathbf{A}(i, j) = \{\{(i, j)\}\}$, then $\mathbf{A}^*(i, j)$ is the set of all minimal arc cut sets for i and j.

A



Part of $\mathbf{A}^*$

$$
\begin{aligned}
\mathbf{A}^*(0,\ 1) &= \{\{(0,1),(2,1)\},\\
&\quad \{(0,1),(0,2),(0,3)\},\\
&\quad \{(0,1),(0,2),(3,2)\}\}\\
\mathbf{A}^*(0,\ 2) &= \{\{(0,2),(1,2),(3,2)\},\\
&\quad \{(0,1),(0,2),(3,2)\},\\
&\quad \{(0,1),(0,2),(0,3)\},\\
&\quad \{(0,2),(0,3),(1,2)\}\}\\
\mathbf{A}^*(2,\ 0) &= \{\{(2,0),(2,1),(3,0)\},\\
&\quad \{(1,0),(2,0),(3,0)\},\\
&\quad \{(1,0),(2,0),(2,3)\},\\
&\{(2,0),(2,1),(2,3)\}\}\\
\mathbf{A}^*(2,\ 3) &= \{\{(2,0),(2,1),(2,3)\},\\
&\quad \{(0,3),(2,3)\},\\
&\quad \{(1,0),(2,0),(2,3)\}\}
\end{aligned}
$$



A Minimax Semiring

$$\text{minimax} \equiv (\mathbb{N}^{\infty},\ \text{min},\ \text{max},\ \infty,\ 0)$$

$$17 \min \infty = 17$$

$$17 \max \infty = \infty$$

How can we interpret this?

$$\mathbf{A}^*(i,\ j) = \min_{p \in \pi(i,\ j)} \max_{(u,v) \in p} \mathbf{A}(u,\ v),$$

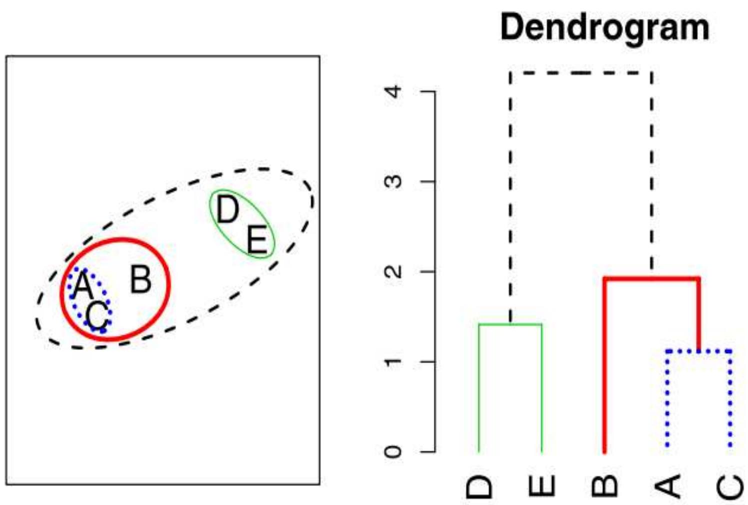
One possible interpretation of Minimax

- Given an adjacency matrix $\mathbf{A}$ over minimax,
- suppose that $\mathbf{A}(i,\ j) = 0 \Leftrightarrow i = j$,
- suppose that $\mathbf{A}$ is symmetric ($\mathbf{A}(i,\ j) = \mathbf{A}(j,\ i)$),
- interpret $\mathbf{A}(i,\ j)$ as measured dissimilarity of i and j,
- interpret $\mathbf{A}^*(i,\ j)$ as inferred dissimilarity of i and j,

Many uses

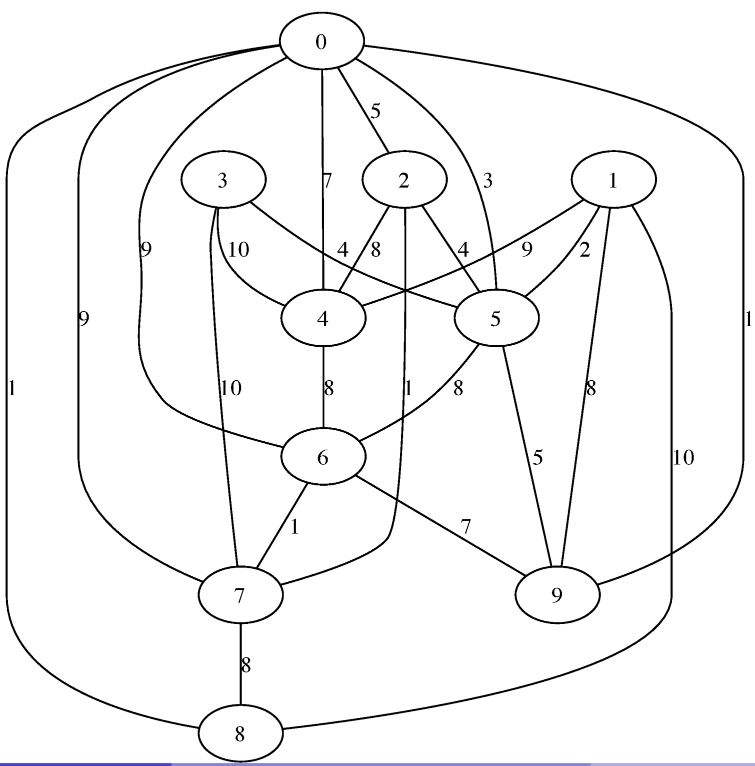
- Hierarchical clustering of large data sets
- Classification in Machine Learning
- Computational phylogenetics
- ...

Dendrograms

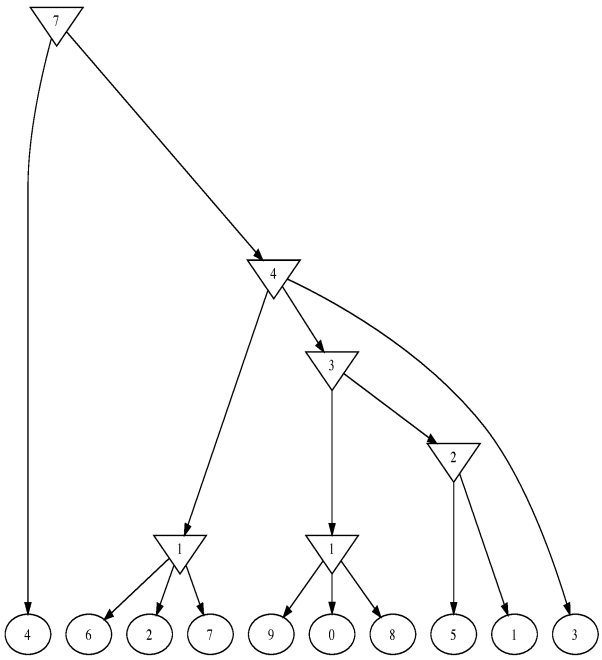


from **Hierarchical Clustering With Prototypes via Minimax Linkage**, Bien and Tibshirani, 2011.

A minimax graph



The solution $\mathbf{A}^*$ drawn as a dendrogram



Hierarchical clustering? Why?

Suppose $(Y, \leqslant, +)$ is a totally ordered with least element 0.

Metric

A metric for set X over $(Y, \leqslant, +)$ is a function $d \in X \times X \to Y$ such that

- $\forall x, y \in X,\ d(x,\ y) = 0 \Leftrightarrow x = y$
- $\forall x, y \in X,\ d(x,\ y) = d(y,\ x)$
- $\forall x, y, z \in X,\ d(x,\ y) \leqslant d(x,\ z) + d(z,\ y)$

Ultrametric

An ultrametric for set X over $(Y, \leqslant)$ is a function $d \in X \times X \to Y$ such that

- $\forall x \in X,\ d(x,\ x) = 0$
- $\forall x, y \in X,\ d(x,\ y) = d(y,\ x)$
- $\forall x, y, z \in X,\ d(x,\ y) \leqslant d(x,\ z) \max d(z,\ y)$

Fun Facts

Fact 5

If $\mathbf{A}$ is an $n \times n$ symmetric $\mathrm{minimax}$ adjacency matrix, then $\mathbf{A}^*$ is a finite ultrametric for $\{0, 1, \ldots, n-1\}$ over $(\mathbb{N}^{\infty}, \leqslant))$.

Fact 6

Suppose each arc weight is unique. Then the set of arcs

$$\{(i, j) \in E \mid \mathbf{A}(i, j) = \mathbf{A}^*(i, j)\}$$

is a minimum spanning tree.

A spanning tree derived from **A** and $\mathbf{A}^*$