Lecture 4: Two interesting Semirings

- Martelli’s semiring for computing minimal cut sets
- The mini-max semiring
Cut Sets

Let $G = (V, E)$ be a directed graph.

- A cut set $C \subseteq E$ for nodes $i$ and $j$ is a set of arcs such that there is no path from $i$ to $j$ in the graph $(V, E - C)$.
- $C$ is minimal if no proper subset of $C$ is an arc cut set.

Martelli's Semiring

Let $G = (V, E)$ be a directed graph.

- $M \equiv (S, \oplus, \otimes, \overline{0}, \overline{T})$
- $S \equiv \{X \in 2^E \mid \forall U, V \in X, U \subseteq V \implies U = V\}$
- $X \oplus Y \equiv$ remove all supersets from $\{U \cup V \mid U \in X, V \in Y\}$
- $X \otimes Y \equiv$ remove all supersets from $X \cup Y$
- $\overline{0} \equiv \{\{}\}$
- $\overline{T} \equiv \{\}$

What does it do?

- If every arc $(i, j)$ has weight $A(i, j) = \{(i, j)\}$, then $A^*(i, j)$ is the set of all minimal arc cut sets for $i$ and $j$. 
Part of $A^*$

$A^*(0, 1) = \{(0, 1), (2, 1)\},$
$\{(0, 1), (0, 2), (0, 3)\},$
$\{(0, 1), (0, 2), (3, 2)\}$

$A^*(0, 2) = \{(0, 2), (1, 2), (3, 2)\},$
$\{(0, 1), (0, 2), (3, 2)\},$
$\{(0, 1), (0, 2), (0, 3)\},$
$\{(0, 2), (0, 3), (1, 2)\}$

$A^*(2, 0) = \{(2, 0), (2, 1), (3, 0)\},$
$\{(1, 0), (2, 0), (3, 0)\},$
$\{(1, 0), (2, 0), (2, 3)\},$
$\{(2, 0), (2, 1), (2, 3)\}$

$A^*(2, 3) = \{(2, 0), (2, 1), (2, 3)\},$
$\{(0, 3), (2, 3)\},$
$\{(1, 0), (2, 0), (2, 3)\}$
A Minimax Semiring

\[ \text{minimax} \equiv (\mathbb{N}^\infty, \min, \max, \infty, 0) \]

\[ 17 \min \infty = 17 \]
\[ 17 \max \infty = \infty \]

How can we interpret this?

\[ A^*(i, j) = \min_{p \in \pi(i, j)} \max_{(u, v) \in p} A(u, v), \]

One possible interpretation of Minimax

- Given an adjacency matrix \( A \) over \( \text{minimax} \),
- suppose that \( A(i, j) = 0 \iff i = j \),
- suppose that \( A \) is symmetric \( (A(i, j) = A(j, i)) \),
- interpret \( A(i, j) \) as measured dissimilarity of \( i \) and \( j \),
- interpret \( A^*(i, j) \) as inferred dissimilarity of \( i \) and \( j \).

Many uses

- Hierarchical clustering of large data sets
- Classification in Machine Learning
- Computational phylogenetics
- ...
Dendrograms

from Hierarchical Clustering With Prototypes via Minimax Linkage, Bien and Tibshirani, 2011.

A minimax graph
Hierarchical clustering? Why?

Suppose \((Y, \leq, +)\) is a totally ordered with least element 0.

**Metric**

A metric for set \(X\) over \((Y, \leq, +)\) is a function \(d : X \times X \to Y\) such that

- \(\forall x, y \in X, \quad d(x, y) = 0 \iff x = y\)
- \(\forall x, y \in X, \quad d(x, y) = d(y, x)\)
- \(\forall x, y, z \in X, \quad d(x, y) \leq d(x, z) + d(z, y)\)

**Ultrametric**

An ultrametric for set \(X\) over \((Y, \leq)\) is a function \(d : X \times X \to Y\) such that

- \(\forall x \in X, \quad d(x, x) = 0\)
- \(\forall x, y \in X, \quad d(x, y) = d(y, x)\)
- \(\forall x, y, z \in X, \quad d(x, y) \leq d(x, z)(\max d(z, y))\)
Fun Facts

Fact 5
If $A$ is an $n \times n$ symmetric minimax adjacency matrix, then $A^*$ is a finite ultrametric for $\{0, 1, \ldots, n-1\}$ over $(\mathbb{N}^\infty, \leq)$.

Fact 6
Suppose each arc weight is unique. Then the set of arcs

$$\{(i, j) \in E \mid A(i, j) = A^*(i, j)\}$$

is a minimum spanning tree.

A spanning tree derived from $A$ and $A^*$