# L11: Algebraic Path Problems with applications to Internet Routing Lecture 4

Timothy G. Griffin

timothy.griffin@cl.cam.ac.uk
Computer Laboratory
University of Cambridge, UK

Michaelmas Term, 2018

# Lecture 4 : Two interesting Semirings

- Martelli's semiring for computing minimal cut sets
- The mini-max semiring

# Cut Sets

Let $G = (V, E)$ be a directed graph.

- A cut set $C \subseteq E$ for nodes $i$ and $j$ is a set of arcs such there is no path from $i$ to $j$ in the graph $(V, E - C)$.
- $C$ is minimal if no proper subset of $C$ is an arc cut set.

# Martelli's Semiring

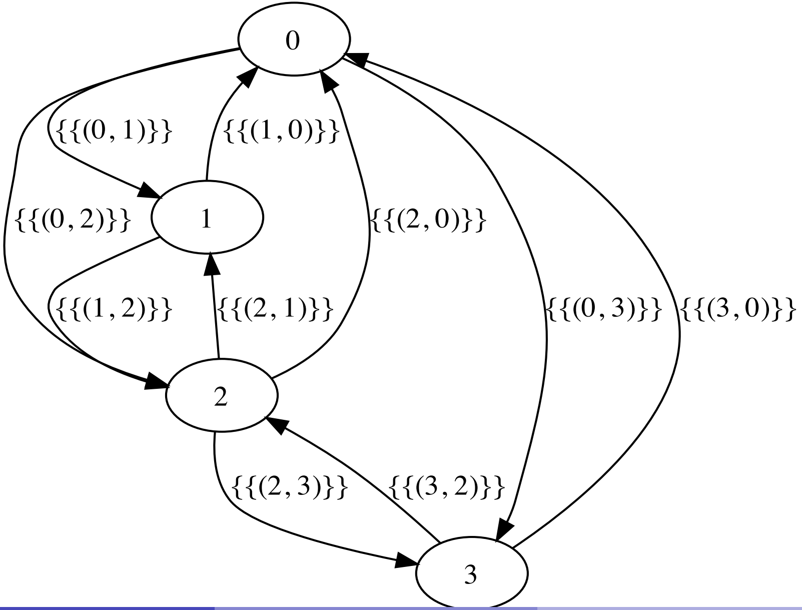
Let $G = (V, E)$ be a directed graph.

$$
\begin{array}{rcl}
\mathbb{M} & \equiv & (S, \oplus, \otimes, \overline{0}, \overline{1}) \\
S & \equiv & \{X \in 2^{2^E} \mid \forall U, V \in X, U \subseteq V \implies U = V\} \\
X \oplus Y & \equiv & \text{remove all supersets from } \{U \cup V \mid U \in X, \ V \in Y\} \\
X \otimes Y & \equiv & \text{remove all supersets from } X \cup Y \\
\overline{0} & \equiv & \{\{\}\} \\
\overline{1} & \equiv & \{\}
\end{array}
$$

## What does it do?

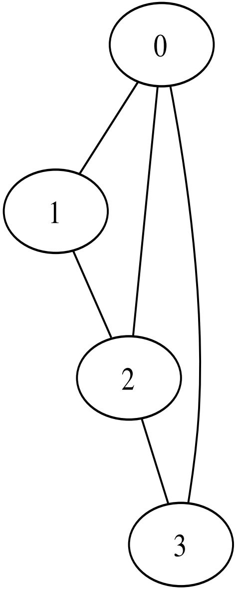
- If every arc $(i, j)$ is has weight $\mathbf{A}(i, j) = \{\{(i, j)\}\}$, then $\mathbf{A}^*(i, j)$ is the set of all minimal arc cut sets for $i$ and $j$.

# A



# Part of $\mathbf{A}^*$

$$
\begin{aligned}
\mathbf{A}^*(0,\ 1) &= \{\{(0,1),(2,1)\},\\
&\quad \{(0,1),(0,2),(0,3)\},\\
&\quad \{(0,1),(0,2),(3,2)\}\}\\
\mathbf{A}^*(0,\ 2) &= \{\{(0,2),(1,2),(3,2)\},\\
&\quad \{(0,1),(0,2),(3,2)\},\\
&\quad \{(0,1),(0,2),(0,3)\},\\
&\quad \{(0,2),(0,3),(1,2)\}\}\\
\mathbf{A}^*(2,\ 0) &= \{\{(2,0),(2,1),(3,0)\},\\
&\quad \{(1,0),(2,0),(3,0)\},\\
&\quad \{(1,0),(2,0),(2,3)\},\\
&\quad \{(2,0),(2,1),(2,3)\}\}\\
\mathbf{A}^*(2,\ 3) &= \{\{(2,0),(2,1),(2,3)\},\\
&\quad \{(0,3),(2,3)\},\\
&\quad \{(1,0),(2,0),(2,3)\}\}
\end{aligned}
$$



# A Minimax Semiring

$$\text{minimax} \equiv (\mathbb{N}^{\infty},\ \text{min},\ \text{max},\ \infty,\ 0)$$

$$17 \min \infty = 17$$

$$17 \max \infty = \infty$$

## How can we interpret this?

$$\mathbf{A}^*(i,\ j) = \min_{p \in \pi(i,\ j)} \max_{(u,v) \in p} \mathbf{A}(u,\ v),$$

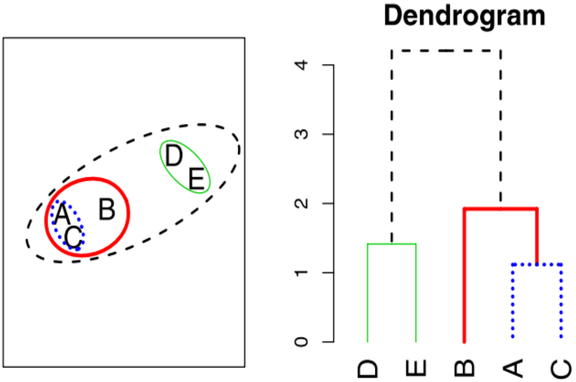
# One possible interpretation of Minimax

- Given an adjacency matrix $\mathbf{A}$ over $\text{minimax}$,
- suppose that $\mathbf{A}(i,\ j) = 0 \Leftrightarrow i = j$,
- suppose that $\mathbf{A}$ is symmetric ($\mathbf{A}(i,\ j) = \mathbf{A}(j,\ i)$),
- interpret $\mathbf{A}(i,\ j)$ as <u>measured</u> dissimilarity of $i$ and $j$,
- interpret $\mathbf{A}^*(i,\ j)$ as <u>inferred</u> dissimilarity of $i$ and $j$,

## Many uses

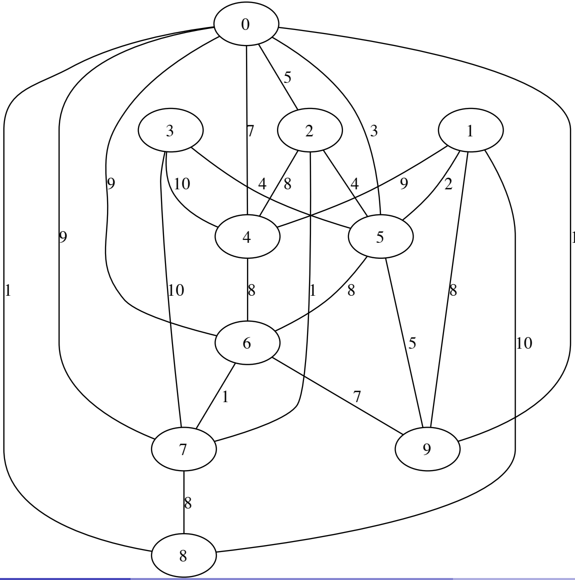
- Hierarchical clustering of large data sets
- Classification in Machine Learning
- Computational phylogenetics
- ...

# Dendrograms

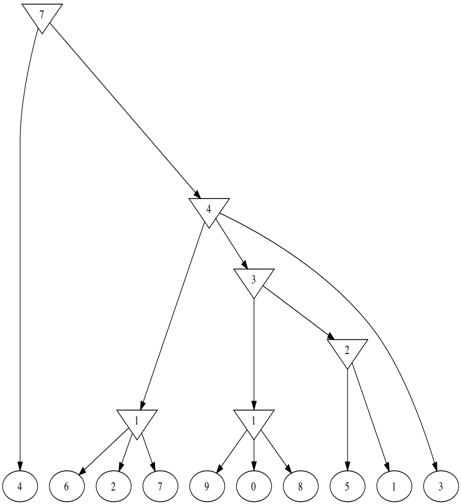



from **Hierarchical Clustering With Prototypes via Minimax Linkage**, Bien and Tibshirani, 2011.

# A minimax graph



# The solution $\mathbf{A}^*$ drawn as a dendrogram



# Hierarchical clustering? Why?

Suppose $(Y, \leqslant, +)$ is a totally ordered with least element 0.

## Metric

A metric for set $X$ over $(Y, \leqslant, +)$ is a function $d \in X \times X \rightarrow Y$ such that

- $\forall x, y \in X, \; d(x, y) = 0 \Leftrightarrow x = y$
- $\forall x, y \in X, \; d(x, y) = d(y, x)$
- $\forall x, y, z \in X, \; d(x, y) \leqslant d(x, z) + d(z, y)$

## Ultrametric

An ultrametric for set $X$ over $(Y, \leqslant)$ is a function $d \in X \times X \rightarrow Y$ such that

- $\forall x \in X, \; d(x, x) = 0$
- $\forall x, y \in X, \; d(x, y) = d(y, x)$
- $\forall x, y, z \in X, \; d(x, y) \leqslant d(x, z) \max d(z, y)$

# Fun Facts

## Fact 5

If $\mathbf{A}$ is an $n \times n$ symmetric $\mathrm{minimax}$ adjacency matrix, then $\mathbf{A}^*$ is a finite ultrametric for $\{0, 1, \ldots, n-1\}$ over $(\mathbb{N}^\infty, \leqslant))$.

## Fact 6

Suppose each arc weight is unique. Then the set of arcs

$$\{(i, j) \in E \mid \mathbf{A}(i, j) = \mathbf{A}^*(i, j)\}$$

is a minimum spanning tree.

# A spanning tree derived from **A** and **A**$^*$