L11: Algebraic Path Problems with applications to Internet Routing
Lecture 14

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Routing vs. Forwarding

- Inspired by the Locator/ID split work
  - See Locator/ID Separation Protocol (LISP)
- Let’s make a distinction between *infrastructure* nodes \( V \) and destinations \( D \).
- Assume \( V \cap D = \{\} \)
- \( M \) is a \( V \times D \) mapping matrix
  - \( M(v, d) \neq \infty \) means that destination (identifier) \( d \) is somehow attached to node (locator) \( v \)
Simple example of forwarding = routing + mapping

Matrix solves

<table>
<thead>
<tr>
<th>Matrix</th>
<th>Solves</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A^*$</td>
<td>$L = (A \times L) \oplus I$</td>
</tr>
<tr>
<td>$A^*M$</td>
<td>$F = (A \times F) \oplus M$</td>
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Forwarding matrix (paths implicit)

$$
M = \begin{bmatrix}
  d_1 & d_2 \\
  1   & \infty & \infty \\
  2   & 3     & \infty \\
  3   & \infty & \infty \\
  4   & \infty & 1    \\
  5   & 2     & 3    \\
\end{bmatrix}
$$

Mapping matrix

$$
F = \begin{bmatrix}
  d_1 & d_2 \\
  1   & 5    & 6    \\
  2   & 3    & 7    \\
  3   & 5    & 5    \\
  4   & 9    & 1    \\
  5   & 2    & 3    \\
\end{bmatrix}
$$
More Interesting Example: “Hot-Potato” Idiom — find attachment that is closest

![Diagram of a network with nodes and edges labeled with distances.]

Mapping matrix

\[ M = \begin{pmatrix}
  d_1 & d_2 \\
  1 & \infty & \infty \\
  2 & (0, 3) & \infty \\
  3 & \infty & \infty \\
  4 & \infty & (0, 1) \\
  5 & (0, 2) & (0, 3)
\end{pmatrix} \]

Forwarding matrix

\[ F = \begin{pmatrix}
  d_1 & d_2 \\
  1 & (2, 3) & (4, 3) \\
  2 & (0, 3) & (4, 3) \\
  3 & (3, 2) & (3, 3) \\
  4 & (7, 2) & (0, 1) \\
  5 & (0, 2) & (0, 3)
\end{pmatrix} \]
General Case

Give $G = (V, E)$

A $|V| \times |V|$ (left) routing matrix $L$ solves an equation of the form

$$L = (A \otimes L) \oplus I,$$

over semiring $S$.

$D$ is the set of destinations.

A $|V| \times |D|$ forwarding matrix is defined as

$$F = L \triangleright M,$$

over some structure $(N, \Box, \triangleright)$, where $\triangleright \in S \to (N \to N)$. 
forwading = path finding + mapping

Does this make sense?

\[ F(i, d) = (L \triangleright M)(i, d) = \Box_{q \in V} L(i, q) \triangleright M(q, d). \]

- Once again we are leaving paths implicit in the construction.
- Routing paths are best paths to egress nodes, selected with respect to \( \Box \)-minimality.
- \( \Box \)-minimality can be very different from selection involved in path finding.
When we are lucky ...

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<td>$A^* \triangleright M$</td>
<td>$F = (A \triangleright F) \square M$</td>
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When does this happen?

When $(N, \square, \triangleright)$ is a (left) semi-module over the semiring $S^a$.

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A model of Internet routing using semi-modules. John N. Billings and Timothy G. Griffin. RelMiCS11/AKA6 2009
(left) Semi-modules

- \((S, \oplus, \otimes, 0, 1)\) is a semiring.

A (left) semi-module over \(S\)

Is a structure \((N, \square, \triangleright, 0_N)\), where

- \((N, \square, 0_N)\) is a commutative monoid
- \(\triangleright\) is a function \(\triangleright \in S \rightarrow (N \rightarrow N)\)
- \((a \otimes b) \triangleright m = a \triangleright (b \triangleright m)\)
- \(\overline{0} \triangleright m = 0_N\)
- \(s \triangleright 0_N = 0_N\)
- \(\overline{1} \triangleright m = m\)

and distributivity holds,

\[
\text{SMLD} : s \triangleright (m \square n) = (s \triangleright m) \square (s \triangleright n)
\]

\[
\text{SMRD} : (s \oplus t) \triangleright m = (s \triangleright m) \square (t \triangleright m)
\]
Example: Hot-Potato

**S idempotent and selective**

\[
\begin{align*}
S &= (S, \oplus_S, \otimes_S) \\
T &= (T, \oplus_T, \otimes_T) \\
\triangleright_{\text{fst}} &\in S \rightarrow (S \times T) \rightarrow (S \times T) \\
S_1 \triangleright_{\text{fst}} (S_2, t) &= (s_1 \otimes_S s_2, t)
\end{align*}
\]

\[
\text{Hot}(S, T) = (S \times T, \vec{x}, \triangleright_{\text{fst}}),
\]

where \(\vec{x}\) is the (left-to-right) lexicographic product of \(\oplus_S\) and \(\oplus_T\).

**Define \(\triangleright_{\text{hp}}\) on matrices**

\[
(L \triangleright_{\text{hp}} M)(i, d) = \vec{x}_{q \in V} \ L(i, q) \triangleright_{\text{fst}} M(q, d)
\]
Example of **hot-potato** routing

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Routing matrix

\[
M = \begin{bmatrix}
\infty & \infty \\
(0, 3) & \infty \\
\infty & \infty \\
(0, 2) & (0, 1) \\
(0, 2) & (0, 3)
\end{bmatrix}
\]

Mapping matrix

\[
F = \begin{bmatrix}
(2, 3) & (4, 3) \\
(0, 3) & (4, 3) \\
(3, 2) & (3, 3) \\
(7, 2) & (0, 1) \\
(0, 2) & (0, 3)
\end{bmatrix}
\]
Example : Cold-Potato

$\DOTimes_T$ selective

\[
S = (S, \DOTimes_S, \Times_S) \\
T = (T, \DOTimes_T, \Times_T) \\
\Ddotimes_{\text{fst}} \in S \to (S \times T) \to (S \times T) \\
s_1 \Ddotimes_{\text{fst}} (s_2, t) = (s_1 \Times_S s_2, t)
\]

\[
\text{Cold}(S, T) = (S \times T, \bar{\times}, \Ddotimes_{\text{fst}})
\]

where $\bar{\times}$ is the (right-to-left) lexicographic product of $\DOTimes_S$ and $\DOTimes_T$.

Define $\Ddotimes_{\text{cp}}$ on matrices

\[
(L \Ddotimes_{\text{cp}} M)(i, d) = \bar{\times}_{q \in V} L(i, q) \Ddotimes_{\text{fst}} M(q, d)
\]
Example of cold-potato routing

\[ \begin{bmatrix} d_1 & d_2 \\ 1 & \infty & \infty \\ 2 & (0, 3) & \infty \\ 3 & \infty & \infty \\ 4 & \infty & (0, 1) \\ 5 & (0, 2) & (0, 3) \end{bmatrix} \]

Mapping matrix

\[ \begin{bmatrix} d_1 & d_2 \\ 1 & (4, 2) & (5, 1) \\ 2 & (4, 2) & (9, 1) \\ 3 & (3, 2) & (4, 1) \\ 4 & (7, 2) & (0, 1) \\ 5 & (0, 2) & (7, 1) \end{bmatrix} \]

Routing matrix

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