L11: Algebraic Path Problems with applications to Internet Routing Lecture 14

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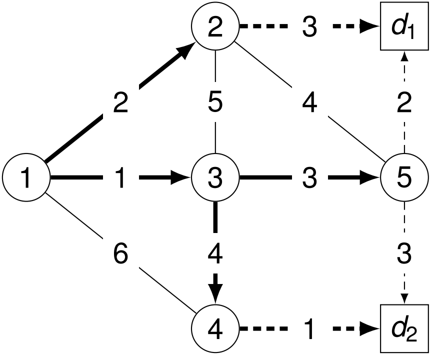
Routing vs. Forwarding

- Inspired by the the Locator/ID split work
  - See Locator/ID Separation Protocol (LISP)
- Let's make a distinction between infrastructure nodes V and destinations D.
- Assume $V \cap D = \{\}$
- $\mathbf{M}$ is a $V \times D$ mapping matrix
  - $\mathbf{M}(v, d) \neq \infty$ means that destination (identifier) d is somehow attached to node (locator) v

Simple example of forwarding = routing + mapping



matrix	solves
$\mathbf{A}^*$	$\mathbf{L} = (\mathbf{A} \otimes \mathbf{L}) \oplus \mathbf{I}$
$\mathbf{A}^*\mathbf{M}$	$\mathbf{F} = (\mathbf{A} \otimes \mathbf{F}) \oplus \mathbf{M}$

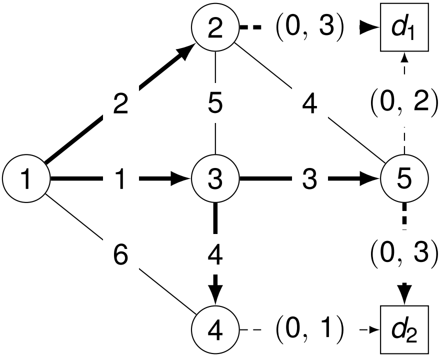
$$\mathbf{M} \quad = \quad \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cc} d_1 & d_2 \end{array} \\ \left[\begin{array}{cc} \infty & \infty \\ 3 & \infty \\ \infty & \infty \\ \infty & 1 \\ 2 & 3 \end{array} \right] \end{array}$$

Mapping matrix

$$\mathbf{F} \quad = \quad \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cc} d_1 & d_2 \end{array} \\ \left[\begin{array}{cc} 5 & 6 \\ 3 & 7 \\ 5 & 5 \\ 9 & 1 \\ 2 & 3 \end{array} \right] \end{array}$$

Forwarding matrix (paths implicit)

More Interesting Example : “Hot-Potato” Idiom — find attachment that is closest



$$
\mathbf{M} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}
\begin{array}{c} \begin{array}{cc} d_1 & d_2 \end{array} \\
\begin{bmatrix} \infty & \infty \\ (0,\ 3) & \infty \\ \infty & \infty \\ \infty & (0,\ 1) \\ (0,\ 2) & (0,\ 3) \end{bmatrix} \end{array}
$$

Mapping matrix

$$
\mathbf{F} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}
\begin{array}{c} \begin{array}{cc} d_1 & d_2 \end{array} \\
\begin{bmatrix} (2,\ 3) & (4,\ 3) \\ (0,\ 3) & (4,\ 3) \\ (3,\ 2) & (3,\ 3) \\ (7,\ 2) & (0,\ 1) \\ (0,\ 2) & (0,\ 3) \end{bmatrix} \end{array}
$$

Forwarding matrix

General Case

Give $G = (V, E)$

A $|V| \times |V|$ (left) routing matrix $\mathbf{L}$ solves an equation of the form

$$\mathbf{L} = (\mathbf{A} \otimes \mathbf{L}) \oplus \mathbf{I},$$

over semiring S.

D is the set of destinations.

A $|V| \times |D|$ forwarding matrix is defined as

$$\mathbf{F} = \mathbf{L} \rhd \mathbf{M},$$

over some structure $(N, \ \square, \ \rhd)$, where $\rhd \in S \to (N \to N)$.

forwading = path finding + mapping

Does this make sense?

$$\mathbf{F}(i,\ d) = (\mathbf{L} \rhd \mathbf{M})(i,\ d) = \Box_{q \in V} \mathbf{L}(i,\ q) \rhd \mathbf{M}(q,\ d).$$

- Once again we are leaving paths implicit in the construction.
- Routing paths are best paths to egress nodes, selected with respect to $\Box$-minimality.
- $\Box$-minimality can be very different from selection involved in path finding.

When we are lucky ...

matrix	solves
$\mathbf{A}^*$	$\mathbf{L} = (\mathbf{A} \otimes \mathbf{L}) \oplus \mathbf{I}$
$\mathbf{A}^* \triangleright \mathbf{M}$	$\mathbf{F} = (\mathbf{A} \triangleright \mathbf{F}) \Box \mathbf{M}$

When does this happen?

When $(N, \Box, \triangleright)$ is a (left) semi-module over the semiring S[a].

[a] A model of Internet routing using semi-modules. John N. Billings and Timothy G. Griffin. RelMiCS11/AKA6 2009

(left) Semi-modules

- $(S, \oplus, \otimes, \overline{0}, \overline{1})$ is a semiring.

A (left) semi-module over S

Is a structure $(N, \square, \rhd, \overline{0}_N)$, where

- $(N, \square, \overline{0}_N)$ is a commutative monoid
- $\rhd$ is a function $\rhd \in S \to (N \to N)$
- $(a \otimes b) \rhd m = a \rhd (b \rhd m)$
- $\overline{0} \rhd m = \overline{0}_N$
- $s \rhd \overline{0}_N = \overline{0}_N$
- $\overline{1} \rhd m = m$

and distributivity holds,

$$\begin{array}{lcrcl} \mathbb{SMLD} & : & s \rhd (m \,\square\, n) & = & (s \rhd m) \,\square\, (s \rhd n) \\ \mathbb{SMRD} & : & (s \oplus t) \rhd m & = & (s \rhd m) \,\square\, (t \rhd m) \end{array}$$

Example : Hot-Potato

S idempotent and selective

$$\begin{array}{rcl} S & = & (S, \oplus_S, \otimes_S) \\ T & = & (T, \oplus_T, \otimes_T) \\ \rhd_{\mathrm{fst}} & \in & S \rightarrow (S \times T) \rightarrow (S \times T) \\ s_1 \rhd_{\mathrm{fst}} (s_2, t) & = & (s_1 \otimes_S s_2, t) \end{array}$$

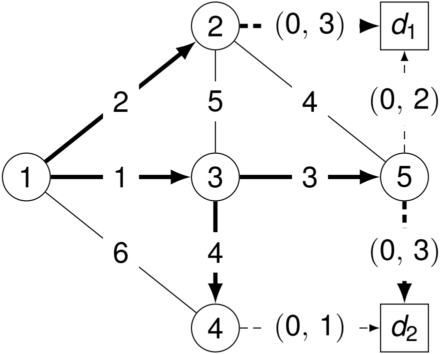
$$\mathrm{Hot}(S, T) = (S \times T, \vec{\times}, \rhd_{\mathrm{fst}}),$$

where $\vec{\times}$ is the (left-to-right) lexicographic product of $\oplus_S$ and $\oplus_T$.

Define $\rhd_{\mathrm{hp}}$ on matrices

$$(\mathbf{L} \rhd_{\mathrm{hp}} \mathbf{M})(i, d) = \vec{\times}_{q \in V} \mathbf{L}(i, q) \rhd_{\mathrm{fst}} \mathbf{M}(q, d)$$

Example of hot-potato routing



$$\mathbf{M} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cc} d_1 & d_2 \end{array} \\ \left[\begin{array}{cc} \infty & \infty \\ (0,\ 3) & \infty \\ \infty & \infty \\ \infty & (0,\ 1) \\ (0,\ 2) & (0,\ 3) \end{array}\right] \end{array}$$

Mapping matrix

$$\mathbf{F} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cc} d_1 & d_2 \end{array} \\ \left[\begin{array}{cc} (2,\ 3) & (4,\ 3) \\ (0,\ 3) & (4,\ 3) \\ (3,\ 2) & (3,\ 3) \\ (7,\ 2) & (0,\ 1) \\ (0,\ 2) & (0,\ 3) \end{array}\right] \end{array}$$

Routing matrix

matrix	solves
$\mathbf{A}^*$	$\mathbf{L} = (\mathbf{A} \otimes \mathbf{L}) \oplus \mathbf{I}$
$\mathbf{A}^* \rhd_{\mathrm{hp}} \mathbf{M}$	$\mathbf{F} = (\mathbf{A} \rhd_{\mathrm{hp}} \mathbf{F}) \vec{\times} \mathbf{M}$

Example : Cold-Potato

$\oplus_T$ selective

$$
\begin{array}{rcl}
S & = & (S,\ \oplus_S,\ \otimes_S) \\
T & = & (T,\ \oplus_T,\ \otimes_T) \\
\rhd_{\mathrm{fst}} & \in & S \to (S \times T) \to (S \times T) \\
s_1 \rhd_{\mathrm{fst}} (s_2,\ t) & = & (s_1 \otimes_S s_2,\ t)
\end{array}
$$

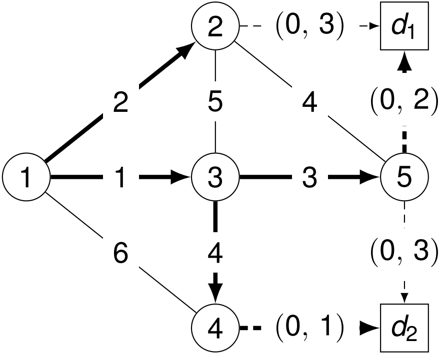
$$
\mathrm{Cold}(S,\ T) = (S \times T,\ \overleftarrow{\times},\ \rhd_{\mathrm{fst}}),
$$

where $\overleftarrow{\times}$ is the (right-to-left) lexicographic product of $\oplus_S$ and $\oplus_T$.

Define $\rhd_{\mathrm{cp}}$ on matrices

$$
(\mathbf{L} \rhd_{\mathrm{cp}} \mathbf{M})(i,\ d) = \overleftarrow{\times}_{q \in V}\ \mathbf{L}(i,\ q) \rhd_{\mathrm{fst}} \mathbf{M}(q,\ d)
$$

Example of cold-potato routing



$$\mathbf{M} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cc} d_1 & d_2 \end{array} \\ \left[\begin{array}{cc} \infty & \infty \\ (0, 3) & \infty \\ \infty & \infty \\ \infty & (0, 1) \\ (0, 2) & (0, 3) \end{array} \right] \end{array}$$

Mapping matrix

$$\mathbf{F} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cc} d_1 & d_2 \end{array} \\ \left[\begin{array}{cc} (4, 2) & (5, 1) \\ (4, 2) & (9, 1) \\ (3, 2) & (4, 1) \\ (7, 2) & (0, 1) \\ (0, 2) & (7, 1) \end{array} \right] \end{array}$$

Routing matrix

matrix	solves
$\mathbf{A}^*$	$\mathbf{L} = (\mathbf{A} \otimes \mathbf{L}) \oplus \mathbf{I}$
$\mathbf{A}^* \rhd_{\mathrm{cp}} \mathbf{M}$	$\mathbf{F} = (\mathbf{A} \rhd_{\mathrm{cp}} \mathbf{F}) \overleftarrow{\times} \mathbf{M}$