Minimal sets

Let $\leq$ be a partial order on $S$.

$$\min(X) \equiv \{ x \in X \mid \forall y \in X, \neg(y < x) \}$$

Define

$$\mathcal{P}_{\text{fin}}(S, \leq) \equiv \{ X \subseteq S \mid X \text{ finite and } \min(X) = X \}.$$ 

and

$$A \cup_{\min} B \equiv \min_{\leq}(A \cup B)$$

Is $\mathcal{P}_{\text{fin}}(S, \leq)$, $\cup_{\min}$ a semigroup?
Is there something more general we can investigate?

If \((S, \cdot)\) is semigroup and \(r \in S \rightarrow S\), then define

\[
\text{reduce}((S, \cdot), r) \equiv (S_r, \cdot_r)
\]

where

\[
S_r \equiv \{ s \in S | r(s) = s \}
\]

\[
x \cdot_r y \equiv r(x \cdot y)
\]

Does \(S_r\) make sense? Think of \(r(x)\) as representing a canonical form for the element \(x\). In that case we want

\[
\text{RIP}(S, r) \equiv \forall x \in S, \ r(r(x)) = r(x)
\]

Call such an \(r\) a reduction on \(S\).

Reduced semigroup?

What about associativity?

\[
\begin{align*}
\text{lhs} & = x \cdot_r (y \cdot_r z) = r(x \cdot r(y \cdot z)) \\
\text{rhs} & = (x \cdot_r y) \cdot_r z = r(r(x \cdot y) \cdot z)
\end{align*}
\]

So we want

\[
\begin{align*}
\text{RIP}(S, r) & \equiv \forall x \in S, \ r(r(x)) = r(x) \\
\text{RAS}((S, \cdot), r) & \equiv \forall x, y, z \in S, \ r(x \cdot r(y \cdot z)) = r(r(x \cdot y) \cdot z)
\end{align*}
\]
Classical Reductions

Wongseelashote 1979

If \((S, \bullet)\) is a semiring and \(r\) is a function from \(S \rightarrow S\), then \(r\) is a (classical) reduction if we have

\[
\begin{align*}
\text{RIP}(S, r) & \equiv \forall x \in S, \ r(r(x)) = r(x) \\
\text{RLC}((S, \bullet), r) & \equiv \forall x, y \in S, \ r(r(x) \bullet y) = r(x \bullet y) \\
\text{RRC}((S, \bullet), r) & \equiv \forall x, y \in S, \ r(x \bullet r(y)) = r(x \bullet y)
\end{align*}
\]

Note that \(\text{RLC}((S, \bullet), r)\) and \(\text{RRC}((S, \bullet), r)\) imply \(\text{RAS}((S, \bullet), r)\).

New Topic!

Properties needed for \((S, \oplus, F)\) to obtain (left) local optima?

Dijkstra's Algorithm requires inflationary

\[
\text{INF} \quad \forall a \in S, \ f \in F : \ a \leq f(a)
\]

Distributed Bellman-Ford (path-vector version) requires strict inflationary

\[
\text{SINF} \quad \forall a \in S, \ F \in F : \ a \neq 0 \implies a < f(a)
\]
Sobrinho’s encoding of the Gao/Rexford rules for BGP

Additive component uses min with
- 0 is the type of a downstream route,
- 1 is the type of a peer route, and
- 2 is the type of an upstream route.
- \(\infty\) is the type of no route.

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This is \(\infty\), but not associative:

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<th>a (\otimes) (b (\otimes) c)</th>
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- Models just the “local preference” component of BGP.
- Can we improve on this with structures of the form \((S, \oplus, F)\)?
Stratified Shortest-Paths Metrics

**Metrics**

\[(s, d) \text{ or } \infty\]

- \(s \not\in \infty\) is a stratum level in \(\{0, 1, 2, \ldots, m-1\}\),
- \(d\) is a “shortest-paths” distance,
- Routing metrics are compared lexicographically

\[(s_1, d_1) < (s_2, d_2) \iff (s_1 < s_2) \lor (s_1 = s_2 \land d_1 < d_2)\]

Stratified Shortest-Paths Policies

**Labels have form** \((f, d)\)

\[(f, d) \triangleright (s, d') \equiv \langle f(s), d + d' \rangle\]

\[(f, d) \triangleright (\infty) \equiv \infty\]

where

\[\langle s, t \rangle = \begin{cases} 
\infty & \text{(if } s = \infty) \\
(s, t) & \text{(otherwise)} 
\end{cases}\]

Yes, a reduction!
Constraint on Policies

\((f, d)\)

- Either \(f\) is inflationary and \(0 < d\),
- or \(f\) is strictly inflationary and \(0 \leq d\).

Why?

\[(\text{SINF}(S) \lor (\text{INF}(S) \land \text{SINF}(T))) \implies \text{SINF}(S \nbigotimes T).\]

Some properties for algebraic structures of the form 
\((S, \oplus, F)\)

<table>
<thead>
<tr>
<th>property</th>
<th>definition</th>
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<tbody>
<tr>
<td>D</td>
<td>(\forall a, b \in S, f \in F : f(a \oplus b) = f(a) \oplus f(b))</td>
</tr>
<tr>
<td>C</td>
<td>(\forall a, b \in S, f \in F : f(a) = f(b) \implies a = b)</td>
</tr>
<tr>
<td>C₀</td>
<td>(\forall a, b \in S, f \in F : f(a) = f(b) \implies (a = b \lor f(a) = \overline{0}))</td>
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<tr>
<td>K</td>
<td>(\forall a, b \in S, f \in F : f(a) = f(b))</td>
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<tr>
<td>K₀</td>
<td>(\forall a, b \in S, f \in F : f(a) \neq f(b) \implies (f(a) = \overline{0} \lor f(b) = \overline{0}))</td>
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All Inflationary Policy Functions for Three Strata

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Both \(D\) and \(C_0\)

This makes combined algebra **distributive**!
Homework 2. Due 16 November.

**Problem 1**
Show that \((P_{\text{fin}}(S, \leq), \cup_{\text{min}})\) is a classical reduction.

**Problem 2**
If \((S, \oplus, \otimes)\) is semiring and \(r \in S \rightarrow S\). Construct \((S_r, \oplus_r, \otimes_r)\) as
\[
S_r = \{s \in S \mid r(s) = s\}
\]
\[
x \oplus_r y = r(x \oplus y)
\]
\[
x \otimes_r y = r(x \otimes y)
\]
Find conditions on \((S, \oplus, \otimes)\) that ensure that we have constructed a semiring.

**Problem 3**
In lecture 9 we “hacked up” an algebraic structure to implement shortest elementary (loop free) paths. Can you use reductions to improve this construction?