# L11: Algebraic Path Problems with applications to Internet Routing Lecture 10

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## Minimal sets

Let  $\leq$  be a partial order on *S*.

$$\min_{\leqslant}(X) \equiv \{x \in X \mid \forall y \in X, \ \neg(y < x)\}$$

Define

$$\mathcal{P}_{\operatorname{fin}}(S, \leqslant) \equiv \{X \subseteq S \mid X \text{ finite and } \min_{\leqslant}(X) = X\}.$$

and

$$A \cup_{\min}^{\leq} B \equiv \min_{\leq} (A \cup B)$$

Is  $(\mathcal{P}_{fin}(\boldsymbol{S}, \leq), \cup_{min}^{\leq})$  a semigroup?

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#### Is there something more general we can investigate?

If  $(S, \bullet)$  is semigroup and  $r \in S \to S$ , then define

reduce
$$((\boldsymbol{S}, \bullet), \boldsymbol{r}) \equiv (\boldsymbol{S}_r, \bullet_r)$$

where

$$S_r \equiv \{s \in S \mid r(s) = s\}$$
  
$$x \bullet_r y \equiv r(x \bullet y)$$

Does  $S_r$  make sense? Think of r(x) as representing a canonical form for the element x. In that case we want

$$\mathbb{RIP}(S, r) \equiv \forall x \in S, r(r(x)) = r(x)$$

Call such an *r* a reduction on *S*.



## Reduced semigroup?

What about associativity?

lhs =  $x \bullet_r (y \bullet_r z) = r(x \bullet r(y \bullet z))$ rhs =  $(x \bullet_r y) \bullet_r z = r(r(x \bullet y) \bullet z)$ 

So we want

$$\mathbb{RIP}(S, r) \equiv \forall x \in S, r(r(x)) = r(x)$$
  
$$\mathbb{RAS}((S, \bullet), r) \equiv \forall x, y, z \in S, r(x \bullet r(y \bullet z)) = r(r(x \bullet y) \bullet z)$$

## **Classical Reductions**

#### Wongseelashote 1979

If  $(S, \bullet)$  is a semiring and *r* is a function from  $S \to S$ , then *r* is a (classical) reduction if we have

 $\begin{array}{rcl} \mathbb{RIP}(S,\ r) &\equiv & \forall x \in S,\ r(r(x)) = r(x) \\ \mathbb{RLC}((S,\ \bullet),\ r) &\equiv & \forall x, y \in S,\ r(r(x) \bullet y) = r(x \bullet y) \\ \mathbb{RRC}((S,\ \bullet),\ r) &\equiv & \forall x, y \in S,\ r(x \bullet r(y)) = r(x \bullet y) \end{array}$ 

Note that  $\mathbb{RLC}((S, \bullet), r)$  and  $\mathbb{RRC}((S, \bullet), r)$  imply  $\mathbb{RAS}((S, \bullet), r)$ .



## New Topic!

Properties needed for  $(S, \oplus, F)$  to obtain (left) local optima?

Dijkstra's Algorithm requires inflationary

 $\mathbb{INF} \quad \forall a \in S, \ f \in F : \ a \leqslant f(a)$ 

Distributed Bellman-Ford (path-vector version) requires strict inflationary

SINF 
$$\forall a \in S, F \in F : a \neq \overline{0} \implies a < f(a)$$

# Sobrinho's encoding of the Gao/Rexford rules for BGP

#### Additive component uses min with

- 0 is the type of a <u>downstream</u> route,
- 1 is the type of a peer route, and
- 2 is the type of an <u>upstream</u> route.
- $\infty$  is the type of no route.

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#### Sobrinho's encoding ...

Multiplicative compone	nt					
	$\otimes$	0	1	2	$\infty$	
	0			$\infty$		
	1	1	$\infty$	$\infty$	$\infty$	
	2	2	2	2	$\infty$	
	$\infty$	$\infty$	$\infty$	$\infty$	$\infty$	

• This is INF, but not associative:

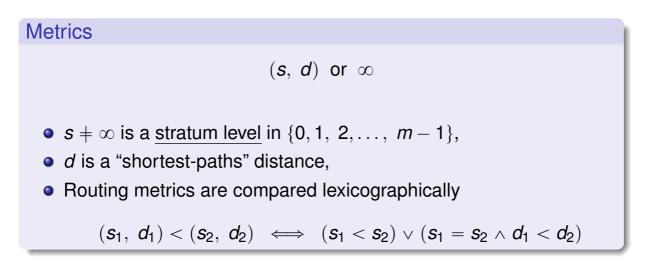
а	b	С	$\boldsymbol{a} \otimes (\boldsymbol{b} \otimes \boldsymbol{c})$	$(\boldsymbol{a} \otimes \boldsymbol{b}) \otimes \boldsymbol{c}$
2	0	1	$\infty$	2
2	0	2	$\infty$	2
2	1	1	$\infty$	2
2	1	2	$\infty$	2

- Models just the "local preference" component of BGP.
- Can we improve on this with structures of the form  $(S, \oplus, F)$ ?

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## **Stratified Shortest-Paths Metrics**



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## **Stratified Shortest-Paths Policies**

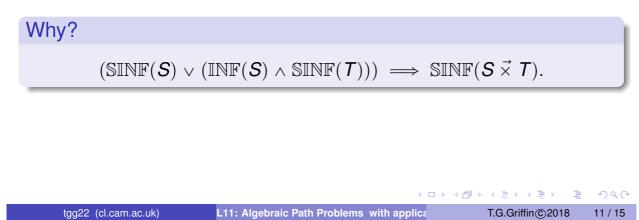
Labels have form 
$$(f, d)$$
  
 $(f, d) \triangleright (s, d') \equiv \langle f(s), d + d' \rangle$   
 $(f, d) \triangleright (\infty) \equiv \infty$   
where  
 $\langle s, t \rangle = \begin{cases} \infty & (\text{if } s = \infty) \\ (s, t) & (\text{otherwise}) \end{cases}$   
Yes, a reduction!

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# **Constraint on Policies**

(*f*, *d*)

- Either *f* is inflationary and 0 < d,
- or *f* is strictly inflationary and  $0 \le d$ .



# Some properties for algebraic structures of the form $(S, \oplus, F)$

property	definition
$\mathbb{D}$	$\forall a, b \in S, f \in F : f(a \oplus b) = f(a) \oplus f(b)$
$\mathbb{C}$	$\forall a, b \in S, f \in F : f(a) = f(b) \implies a = b$
$\mathbb{C}_{\overline{0}}$	$\forall a, b \in S, \ f \in F \ : \ f(a) = f(b) \implies (a = b \lor f(a) = \overline{0})$
$\mathbb{K}$	$\forall a, b \in S, f \in F : f(a) = f(b)$
$\mathbb{K}_{\overline{0}}$	$\forall a, b \in S, \ f \in F : \ f(a) \neq f(b) \implies (f(a) = \overline{0} \lor f(b) = \overline{0})$

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# All Inflationary Policy Functions for Three Strata

	0	1	2	D	$\mathbb{C}^\infty$	$\mathbb{K}^{\infty}$		0	1	2	D	$\mathbb{C}^{\infty}$	$\mathbb{K}^{\infty}$
а	0	1	2	*	*		m	2	1	2			
b	0	1	$\infty$	*	*		n	2	1	$\infty$		*	
С	0	2	2	*			ο	2	2	2	*		*
d	0	2	$\infty$	*	*		р	2	2	$\infty$	*		*
е	0	$\infty$	2		*		q	2	$\infty$	2			*
f	0	$\infty$	$\infty$	*	*	*	r	2	$\infty$	$\infty$	*	*	*
g	1	1	2	*			S	$\infty$	1	2		*	
h	1	1	$\infty$	*		*	t	$\infty$	1	$\infty$		*	*
i	1	2	2	*			u	$\infty$	2	2			*
j	1	2	$\infty$	*	*		v	$\infty$	2	$\infty$		*	*
k	1	$\infty$	2		*		w	$\infty$	$\infty$	2		*	*
Ι	1	$\infty$	$\infty$	*	*	*	X	$\infty$	$\infty$	$\infty$	*	*	*
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# Both $\mathbb D$ and $\mathbb C_{\overline 0}$

This makes combined alge	ebra	a di	istril	butiv	ve!
		0	1	2	
	a	0	1	2	
	b	0	1	$\infty$	
	d	0	2	$\infty$	
	f	0	$\infty$	$\infty$	
	j	1	2	$\infty$	
	I	1	$\infty$	$\infty$	
	r	2	$\infty$	$\infty$	
	x	$\infty$	$\infty$	$\infty$	

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## Homework 2. Due 16 November.

#### Problem 1

Show that  $(\mathcal{P}_{fin}(S, \leq), \cup_{min}^{\leq})$  is a classical reduction.

#### Problem 2

If  $(S, \oplus, \otimes)$  is semiring and  $r \in S \to S$ . Construct  $(S_r, \oplus_r, \otimes_r)$  as

$$x \oplus_r y = r(x \oplus y)$$

$$x \otimes_r y = r(x \otimes y)$$

Find conditions on  $(S, \oplus, \otimes)$  that ensure that we have constructed a semiring.

#### **Problem 3**

In lecture 9 we "hacked up" an algebraic structure to implement shortest elementary (loop free) paths. Can you use reductions to improve this construction?

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