L11: Algebraic Path Problems with applications to Internet Routing Lecture 10

Timothy G. Griffin

timothy.griffin@cl.cam.ac.uk
Computer Laboratory
University of Cambridge, UK

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Minimal sets

Let $\leqslant$ be a partial order on S.

$$\min_{\leqslant}(X) \equiv \{x \in X \mid \forall y \in X,\ \neg(y < x)\}$$

Define

$$\mathcal{P}_{\text{fin}}(S,\ \leqslant) \equiv \{X \subseteq S \mid X \text{ finite and } \min_{\leqslant}(X) = X\}.$$

and

$$A \cup_{\min}^{\leqslant} B \quad \equiv \quad \min_{\leqslant}(A \cup B)$$

Is $(\mathcal{P}_{\text{fin}}(S,\ \leqslant),\ \cup_{\min}^{\leqslant})$ a semigroup?

Is there something more general we can investigate?

If $(S, \bullet)$ is semigroup and $r \in S \to S$, then define

$$\text{reduce}((S, \bullet), r) \equiv (S_r, \bullet_r)$$

where

$$\begin{array}{rcl} S_r & \equiv & \{s \in S \mid r(s) = s\} \\ x \bullet_r y & \equiv & r(x \bullet y) \end{array}$$

Does S_r make sense? Think of $r(x)$ as representing a canonical form for the element x. In that case we want

$$\mathbb{RIP}(S, r) \equiv \forall x \in S, r(r(x)) = r(x)$$

Call such an r a reduction on S.

Reduced semigroup?

What about associativity?

$$\begin{array}{lcll} \text{lhs} & = & x \bullet_r (y \bullet_r z) & = r(x \bullet r(y \bullet z)) \\ \text{rhs} & = & (x \bullet_r y) \bullet_r z & = r(r(x \bullet y) \bullet z) \end{array}$$

So we want

$$\begin{array}{rcl} \mathbb{RIP}(S,\ r) & \equiv & \forall x \in S,\ r(r(x)) = r(x) \\ \mathbb{RAS}((S,\ \bullet),\ r) & \equiv & \forall x, y, z \in S,\ r(x \bullet r(y \bullet z)) = r(r(x \bullet y) \bullet z) \end{array}$$

Classical Reductions

Wongseelashote 1979

If $(S, \bullet)$ is a semiring and r is a function from $S \rightarrow S$, then r is a (classical) reduction if we have

$$\begin{array}{rcl} \mathbb{RIP}(S,\ r) & \equiv & \forall x \in S,\ r(r(x)) = r(x) \\ \mathbb{RLC}((S,\ \bullet),\ r) & \equiv & \forall x, y \in S,\ r(r(x) \bullet y) = r(x \bullet y) \\ \mathbb{RRC}((S,\ \bullet),\ r) & \equiv & \forall x, y \in S,\ r(x \bullet r(y)) = r(x \bullet y) \end{array}$$

Note that $\mathbb{RLC}((S,\ \bullet),\ r)$ and $\mathbb{RRC}((S,\ \bullet),\ r)$ imply $\mathbb{RAS}((S,\ \bullet),\ r)$.

New Topic!

Properties needed for $(S, \oplus, F)$ to obtain (left) local optima?

Dijkstra's Algorithm requires inflationary

$$\mathbb{INF} \quad \forall a \in S,\ f \in F \ :\ a \leqslant f(a)$$

Distributed Bellman-Ford (path-vector version) requires strict inflationary

$$\mathbb{SINF} \quad \forall a \in S,\ F \in F \ :\ a \neq \overline{0} \implies a < f(a)$$

Sobrinho's encoding of the Gao/Rexford rules for BGP

Additive component uses min with

- 0 is the type of a downstream route,
- 1 is the type of a peer route, and
- 2 is the type of an upstream route.
- ∞ is the type of no route.

Sobrinho's encoding ...

Multiplicative component

$\otimes$	0	1	2	∞
0	0	∞	∞	∞
1	1	∞	∞	∞
2	2	2	2	∞
∞	∞	∞	∞	∞

- This is $\mathbb{INF}$, but not associative:

a	b	c	$a \otimes (b \otimes c)$	$(a \otimes b) \otimes c$
2	0	1	∞	2
2	0	2	∞	2
2	1	1	∞	2
2	1	2	∞	2

- Models just the "local preference" component of BGP.
- Can we improve on this with structures of the form $(S, \oplus, F)$?

Stratified Shortest-Paths Metrics

Metrics

$$(s,\ d)\ \text{ or }\ \infty$$

- $s \neq \infty$ is a stratum level in $\{0, 1,\ 2, \ldots,\ m-1\}$,
- d is a “shortest-paths” distance,
- Routing metrics are compared lexicographically

$$(s_1,\ d_1) < (s_2,\ d_2) \iff (s_1 < s_2) \vee (s_1 = s_2 \wedge d_1 < d_2)$$

Stratified Shortest-Paths Policies

Labels have form $(f,\ d)$

$$
\begin{aligned}
(f,\ d) \rhd (s,\ d') &\equiv \langle f(s),\ d + d' \rangle \\
(f,\ d) \rhd (\infty) &\equiv \infty
\end{aligned}
$$

where

$$
\langle s,\ t \rangle = \begin{cases} \infty & (\text{if } s = \infty) \\ (s,\ t) & (\text{otherwise}) \end{cases}
$$

Yes, a reduction!

Constraint on Policies

$$(f,\ d)$$

- Either f is inflationary and $0 < d$,
- or f is strictly inflationary and $0 \leqslant d$.

Why?

$$(\mathbb{SINF}(S) \vee (\mathbb{INF}(S) \wedge \mathbb{SINF}(T))) \implies \mathbb{SINF}(S \vec{\times} T).$$

Some properties for algebraic structures of the form $(S, \oplus, F)$

property	definition
$\mathbb{D}$	$\forall a, b \in S,\ f \in F\ :\ f(a \oplus b) = f(a) \oplus f(b)$
$\mathbb{C}$	$\forall a, b \in S,\ f \in F\ :\ f(a) = f(b) \implies a = b$
$\mathbb{C}_{\overline{0}}$	$\forall a, b \in S,\ f \in F\ :\ f(a) = f(b) \implies (a = b \vee f(a) = \overline{0})$
$\mathbb{K}$	$\forall a, b \in S,\ f \in F\ :\ f(a) = f(b)$
$\mathbb{K}_{\overline{0}}$	$\forall a, b \in S,\ f \in F\ :\ f(a) \neq f(b) \implies (f(a) = \overline{0} \vee f(b) = \overline{0})$

All Inflationary Policy Functions for Three Strata

	0	1	2	D	$\mathbb{C}_\infty$	$\mathbb{K}_\infty$
a	0	1	2	⋆	⋆	
b	0	1	∞	⋆	⋆	
c	0	2	2	⋆		
d	0	2	∞	⋆	⋆	
e	0	∞	2		⋆	
f	0	∞	∞	⋆	⋆	⋆
g	1	1	2	⋆		
h	1	1	∞	⋆		⋆
i	1	2	2	⋆		
j	1	2	∞	⋆	⋆	
k	1	∞	2		⋆	
l	1	∞	∞	⋆	⋆	⋆

	0	1	2	D	$\mathbb{C}_\infty$	$\mathbb{K}_\infty$
m	2	1	2			
n	2	1	∞		⋆	
o	2	2	2	⋆		⋆
p	2	2	∞	⋆		⋆
q	2	∞	2			⋆
r	2	∞	∞	⋆	⋆	⋆
s	∞	1	2		⋆	
t	∞	1	∞		⋆	⋆
u	∞	2	2			⋆
v	∞	2	∞		⋆	⋆
w	∞	∞	2		⋆	⋆
x	∞	∞	∞	⋆	⋆	⋆

Both $\mathbb{D}$ and $\mathbb{C}_{\overline{0}}$

This makes combined algebra distributive!

	0	1	2
a	0	1	2
b	0	1	∞
d	0	2	∞
f	0	∞	∞
j	1	2	∞
l	1	∞	∞
r	2	∞	∞
x	∞	∞	∞

Homework 2. Due 16 November.

Problem 1

Show that $(\mathcal{P}_{\text{fin}}(S, \leqslant), \cup_{\min}^{\leqslant})$ is a classical reduction.

Problem 2

If $(S, \oplus, \otimes)$ is semiring and $r \in S \to S$. Construct $(S_r, \oplus_r, \otimes_r)$ as

1. $S_r = \{s \in S \mid r(s) = s\}$
2. $x \oplus_r y = r(x \oplus y)$
3. $x \otimes_r y = r(x \otimes y)$

Find conditions on $(S, \oplus, \otimes)$ that ensure that we have constructed a semiring.

Problem 3

In lecture 9 we “hacked up” an algebraic structure to implement shortest elementary (loop free) paths. Can you use reductions to improve this construction?