Minimal sets

Let $\leq$ be a partial order on $S$.

$$\min(\{x \in X \mid \forall y \in X, \neg(y \lessdot x)\})$$

Define

$$\mathcal{P}_{\text{fin}}(S, \leq) \equiv \{X \subseteq S \mid X \text{ finite and } \min(\leq) = X\}.$$ 

and

$$A \cup_{\min} B \equiv \min_{\leq}(A \cup B)$$

Is $(\mathcal{P}_{\text{fin}}(S, \leq), \cup_{\min})$ a semigroup?
Is there something more general we can investigate?

If \((S, \bullet)\) is semigroup and \(r \in S \rightarrow S\), then define

\[
\text{reduce}((S, \bullet), r) \equiv (S_r, \bullet_r)
\]

where

\[
S_r \equiv \{ s \in S \mid r(s) = s \}
\]

\[
x \bullet_r y \equiv r(x \bullet y)
\]

Does \(S_r\) make sense? Think of \(r(x)\) as representing a canonical form for the element \(x\). In that case we want

\[
\text{RIP}(S, r) \equiv \forall x \in S, \ r(r(x)) = r(x)
\]

Call such an \(r\) a **reduction** on \(S\).
Reduced semigroup?

What about associativity?

\[
\text{lhs} = x \cdot_r (y \cdot_r z) = r(x \cdot r(y \cdot z))
\]
\[
\text{rhs} = (x \cdot_r y) \cdot_r z = r(r(x \cdot y) \cdot z)
\]

So we want

\[
\text{RIP}(S, r) \equiv \forall x \in S, \ r(r(x)) = r(x)
\]
\[
\text{RAS}((S, \cdot), r) \equiv \forall x, y, z \in S, \ r(x \cdot r(y \cdot z)) = r(r(x \cdot y) \cdot z)
\]
Classical Reductions

Wongseelahote 1979

If $(S, \bullet)$ is a semiring and $r$ is a function from $S \rightarrow S$, then $r$ is a (classical) reduction if we have

\[
\begin{align*}
\text{RIP}(S, r) & \equiv \forall x \in S, \ r(r(x)) = r(x) \\
\text{RLC}((S, \bullet), r) & \equiv \forall x, y \in S, \ r(r(x) \bullet y) = r(x \bullet y) \\
\text{RRC}((S, \bullet), r) & \equiv \forall x, y \in S, \ r(x \bullet r(y)) = r(x \bullet y)
\end{align*}
\]

Note that $\text{RLC}((S, \bullet), r)$ and $\text{RRC}((S, \bullet), r)$ imply $\text{RAS}((S, \bullet), r)$. 
Properties needed for \((S, \oplus, F)\) to obtain (left) local optima?

**Dijkstra’s Algorithm requires inflationary**

\[
\inf \quad \forall a \in S, f \in F : a \leq f(a)
\]

**Distributed Bellman-Ford (path-vector version) requires strict inflationary**

\[
\inf\inf \quad \forall a \in S, F \in F : a \neq 0 \implies a < f(a)
\]
Sobrinho’s encoding of the Gao/Rexford rules for BGP

Additive component uses \( \min \) with

- 0 is the type of a **downstream** route,
- 1 is the type of a **peer** route, and
- 2 is the type of an **upstream** route.
- \( \infty \) is the type of no route.
Sobrinho’s encoding ...

Multiplicative component

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<tr>
<th>⊗</th>
<th>0</th>
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</table>

This is **INF**, but not associative:

<table>
<thead>
<tr>
<th>a</th>
<th>b</th>
<th>c</th>
<th>a ⊗ (b ⊗ c)</th>
<th>(a ⊗ b) ⊗ c</th>
</tr>
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</table>

- Models just the “local preference” component of BGP.
- Can we improve on this with structures of the form \((S, \oplus, F)\)?
Stratified Shortest-Paths Metrics

Metrics

\[(s, d) \text{ or } \infty\]

- \(s \neq \infty\) is a stratum level in \(\{0, 1, 2, \ldots, m - 1\}\),
- \(d\) is a “shortest-paths” distance,
- Routing metrics are compared lexicographically

\[(s_1, d_1) < (s_2, d_2) \iff (s_1 < s_2) \lor (s_1 = s_2 \land d_1 < d_2)\]
Stratified Shortest-Paths Policies

Labels have form \((f, d)\)

\[(f, d) \triangleright (s, d') \equiv \langle f(s), d + d' \rangle\]

\[(f, d) \triangleright (\infty) \equiv \infty\]

where

\[\langle s, t \rangle = \begin{cases} 
\infty & \text{(if } s = \infty) \\
(s, t) & \text{(otherwise)}
\end{cases}\]

Yes, a reduction!
Constraint on Policies

Either $f$ is inflationary and $0 < d$,
or $f$ is strictly inflationary and $0 \leq d$.

Why?

\[(\text{SINF}(S) \lor (\text{INF}(S) \land \text{SINF}(T)))) \implies \text{SINF}(S \rightarrow T)\]
Some properties for algebraic structures of the form $(S, \oplus, F)$

<table>
<thead>
<tr>
<th>property</th>
<th>definition</th>
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<tr>
<td>$D$</td>
<td>$\forall a, b \in S, f \in F : f(a \oplus b) = f(a) \oplus f(b)$</td>
</tr>
<tr>
<td>$C$</td>
<td>$\forall a, b \in S, f \in F : f(a) = f(b) \implies a = b$</td>
</tr>
<tr>
<td>$C_0$</td>
<td>$\forall a, b \in S, f \in F : f(a) = f(b) \implies (a = b \lor f(a) = \bar{0})$</td>
</tr>
<tr>
<td>$K$</td>
<td>$\forall a, b \in S, f \in F : f(a) = f(b)$</td>
</tr>
<tr>
<td>$K_0$</td>
<td>$\forall a, b \in S, f \in F : f(a) \neq f(b) \implies (f(a) = \bar{0} \lor f(b) = \bar{0})$</td>
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### All Inflationary Policy Functions for Three Strata

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Both $\mathbb{D}$ and $\mathbb{C}_0$ make combined algebra distributive!

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Problem 1

Show that \((\mathcal{P}_{\text{fin}}(S, \leq), \cup_{\text{min}})\) is a classical reduction.

Problem 2

If \((S, \oplus, \otimes)\) is semiring and \(r \in S \rightarrow S\). Construct \((S_r, \oplus_r, \otimes_r)\) as

1. \(S_r = \{s \in S \mid r(s) = s\}\)
2. \(x \oplus_r y = r(x \oplus y)\)
3. \(x \otimes_r y = r(x \otimes y)\)

Find conditions on \((S, \oplus, \otimes)\) that ensure that we have constructed a semiring.

Problem 3

In lecture 9 we “hacked up” an algebraic structure to implement shortest elementary (loop free) paths. Can you use reductions to improve this construction?