Lecture 16

Recall (for a small category $\mathbf{C}$):

Yoneda functor $\mathbf{y} : \mathbf{C} \to \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$

maps each $X \in \mathbf{C}$ to $\mathbf{C}(\_, X) : \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$ and each $X \xrightarrow{f} Y$ in $\mathbf{C}$ to the natural transformation $\mathbf{C}(\_, X) \to \mathbf{C}(\_, Y)$ whose component at each $Z \in \mathbf{C}$ is the function $f_* : \mathbf{C}(Z, X) \to \mathbf{C}(Z, Y)$ mapping g to $f \circ g$.

Recall (for a small category $\mathbf{C}$):

Yoneda functor $\mathbf{y} : \mathbf{C} \to \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$

Yoneda Lemma: there is a bijection $\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}(\mathbf{y}X, F) \cong F(X)$ which is natural both in $F \in \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$ and $X \in \mathbf{C}$.

In particular, $\mathbf{y}$ is a full and faithful functor.

Hence $\mathbf{y}X \cong \mathbf{y}Y \;\Rightarrow\; X \cong Y$, i.e.

Given $\mathbf{C}$-objects X, Y, to show $X \cong Y$ in $\mathbf{C}$, it suffices to give bijections $\mathbf{C}(Z, X) \cong \mathbf{C}(Z, Y)$ in $\mathbf{Set}$ that are natural in $Z \in \mathtt{obj}\,\mathbf{C}$,

Recall (for a small category $\mathbf{C}$):

Yoneda functor $\mathbf{y} : \mathbf{C} \to \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$

Yoneda Lemma: there is a bijection $\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}(\mathbf{y}X, F) \cong F(X)$ which is natural both in $F \in \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$ and $X \in \mathbf{C}$.

In particular, $\mathbf{y}$ is a full and faithful functor.

Hence $\mathbf{y}X \cong \mathbf{y}Y \;\Rightarrow\; X \cong Y$, i.e.

> Given $\mathbf{C}$-objects X, Y, to show $X \cong Y$ in $\mathbf{C}$, it suffices to give bijections $\mathbf{C}(Z, X) \cong \mathbf{C}(Z, Y)$ in $\mathbf{Set}$ that are natural in $Z \in \mathtt{obj}\,\mathbf{C}$,
> or dually, bijections $\mathbf{C}(X, Z) \cong \mathbf{C}(Y, Z)$ in $\mathbf{Set}$ that are natural in $Z \in \mathtt{obj}\,\mathbf{C}$.

E.g. in a ccc that has binary coproducts, one can use this to show $(Y + Z) \times X \cong (Y \times X) + (Z \times X)$ ...

Recall (for a small category $\mathbf{C}$):

Yoneda functor $\mathbf{y} : \mathbf{C} \to \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$

Yoneda Lemma: there is a bijection $\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}(\mathbf{y}X, F) \cong F(X)$ which is natural both in $F \in \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$ and $X \in \mathbf{C}$.

An application of the Yoneda Lemma:

Theorem. For each small category $\mathbf{C}$, the category $\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$ of presheaves is cartesian closed.

Theorem. For each small category $\mathbf{C}$, the category $\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$ of presheaves is cartesian closed.

Theorem. For each small category $\mathbf{C}$, the category $\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$ of presheaves is cartesian closed.

Proof sketch.

Terminal object in $\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$ is the functor $1 : \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$ given by

$$\begin{cases} 1(X) \triangleq \{0\} & \text{terminal object in } \mathbf{Set} \\ 1(f) \triangleq \mathtt{id}_{\{0\}} \end{cases}$$

Theorem. For each small category $\mathbf{C}$, the category $\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$ of presheaves is cartesian closed.

Proof sketch.

Product of $F, G \in \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$ is the functor $F \times G : \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$ given by

$$\begin{cases} (F \times G)(X) \triangleq F(X) \times G(X) & \text{cartesian product of sets} \\ (F \times G)(f) \triangleq F(f) \times G(f) \end{cases}$$

with projection morphisms $F \xleftarrow{\pi_1} F \times G \xrightarrow{\pi_2} G$ given by the natural transformations whose components at $X \in \mathbf{C}$ are the projection functions $F(X) \xleftarrow{\pi_1} F(X) \times G(X) \xrightarrow{\pi_2} G(X)$.

Theorem. For each small category $\mathbf{C}$, the category $\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$ of presheaves is cartesian closed.

Proof sketch.

We can work out what the value of the exponential $G^F \in \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$ at $X \in \mathbf{C}$ has to be using the Yoneda Lemma:

$$G^F(X) \underset{\boxed{\text{Yoneda Lemma}}}{\cong} \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}(\mathrm{y}X, G^F) \underset{\boxed{\text{universal property of the exponential}}}{\cong} \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}(\mathrm{y}X \times F, G)$$

Theorem. For each small category $\mathbf{C}$, the category $\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$ of presheaves is cartesian closed.

Proof sketch.

We can work out what the value of the exponential $G^F \in \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$ at $X \in \mathbf{C}$ has to be using the Yoneda Lemma:

$$G^F(X) \cong \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}(\mathrm{y}X, G^F) \cong \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}(\mathrm{y}X \times F, G)$$

We take the set $\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}(\mathrm{y}X \times F, G)$ to be the definition of the value of G^F at X...

Exponential objects in $\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$:

$$\boxed{G^F(X) \triangleq \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}(\mathrm{y}X \times F, G)}$$

Given $Y \xrightarrow{f} X$ in $\mathbf{C}$, we have $\mathrm{y}Y \xrightarrow{\mathrm{y}f} \mathrm{y}X$ in $\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$ and hence

$$\begin{array}{rcl} G^F(Y) \triangleq \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}(\mathrm{y}Y \times F, G) & \to & \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}(\mathrm{y}X \times F, G) \triangleq G^F(X) \\ \theta & \mapsto & \theta \circ (\mathrm{y}f \times \mathtt{id}_F) \end{array}$$

We define

$$\boxed{G^F(f) \triangleq (\mathrm{y}f \times \mathtt{id}_F)^*}$$

Have to check that these definitions make G^F ino a functor $\mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$.

Application morphisms in $\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$:

Given $F, G \in \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$, the application morphism

$$\mathtt{app} : G^F \times F \to G$$

is the natural transformation whose component at $X \in \mathbf{C}$ is the function

$$(G^F \times F)(X) \triangleq G^F(X) \times F(X) \triangleq \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}(\mathrm{y}X \times F, G) \times F(X) \xrightarrow{\mathtt{app}_X} G(X)$$

defined by

$$\boxed{\mathtt{app}_X(\theta, x) \triangleq \theta_X(\mathtt{id}_X, x)}$$

Have to check that this is natural in X.

Currying operation in $\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$:

$$\left(H \times F \xrightarrow{\theta} G\right) \mapsto \left(H \xrightarrow{\mathrm{cur}\,\theta} G^F\right)$$

Given $H \times F \xrightarrow{\theta} G$ in $\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$, the component of $\mathrm{cur}\,\theta$ at $X \in \mathbf{C}$

$$H(X) \xrightarrow{(\mathrm{cur}\,\theta)_X} G^F(X) \triangleq \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}(\mathbf{y}X \times F, G)$$

is the function mapping each $z \in H(X)$ to the natural transformation $\mathbf{y}X \times F \to G$ whose component at $Y \in \mathbf{C}$ is the function

$$(\mathbf{y}X \times F)(Y) \triangleq \mathbf{C}(Y, X) \times F(Y) \to G(Y)$$

defined by

$$\boxed{((\mathrm{cur}\,\theta)_X(z))_Y(f, y) \triangleq \theta_Y(H(f)(z), y)}$$

Currying operation in $\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$:

$$\left(H \times F \xrightarrow{\theta} G\right) \mapsto \left(H \xrightarrow{\mathtt{cur}\,\theta} G^F\right)$$

$$\boxed{((\mathtt{cur}\,\theta)_X(z))_Y(f, y) \triangleq \theta_Y(H(f)(z), y)}$$

Have to check that this is natural in Y,

then that $(\mathtt{cur}\,\theta)_X$ is natural in X,

then that $\mathtt{cur}\,\theta$ is the unique morphism $H \xrightarrow{\varphi} G^F$ in $\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$ satisfying $\mathtt{app} \circ (\varphi \times \mathtt{id}_F) = \theta$.

Theorem. For each small category $\mathbf{C}$, the category $\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$ of presheaves is cartesian closed.

So we can interpret simply typed lambda calculus in any presheaf category.

More than that, presheaf categories (usefully) model dependently-typed languages.

Next steps in basic category theory

- equivalence of categories
- limits and colimits of diagrams in categories
- (co)monads and their (co)algebras

Some current themes involving category theory

- semantics of effects & co-effects in programming languages

  (monads and comonads)

- homotopy type theory

  (higher-dimensional category theory)

- structural aspects of networks, quantum computation/protocols, ...

  (string diagrams for monoidal categories)