Lecture 15
Presheaf categories

Let $\mathbf{C}$ be a small category. The functor category $\text{Set}^{\mathbf{C}^{\text{op}}}$ is called the category of presheaves on $\mathbf{C}$.

- objects are contravariant functors from $\mathbf{C}$ to $\text{Set}$
- morphisms are natural transformations

Much used in the semantics of various dependently-typed languages and logics.
Yoneda functor

\[ y : C \to \text{Set}^{\text{C}^{\text{op}}} \]

(where \( C \) is a small category)

is the Curried version of the \( \text{hom} \) functor

\[
C \times C^{\text{op}} \cong C^{\text{op}} \times C \xrightarrow{\text{Hom}_C} \text{Set}
\]
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\[ C \times C^{\text{op}} \cong C^{\text{op}} \times C \xrightarrow{\text{Hom}_C} \text{Set} \]

For each \( C \)-object \( X \), the object \( yX \in \text{Set}^{\text{C}^\text{op}} \) is the functor \( C(\_, X) : C^{\text{op}} \to \text{Set} \) given by

\[
\begin{align*}
Z & \mapsto C(Z, X) & g \circ f \\
\downarrow f & \mapsto \downarrow & \uparrow \\
Y & \mapsto C(Y, X) & \uparrow g
\end{align*}
\]
Yoneda functor

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\begin{align*}
Z & \mapsto C(Z, X) \\
Y & \mapsto C(Y, X)
\end{align*}
\]

\[ g \circ f \]

\[ f^* \]

this function is often written as \( f^* \)
Yoneda functor

\[ y : C \to \text{Set}^{\text{C}^{\text{op}}} \]

(where \( C \) is a small category)

is the Curried version of the hom functor

\[ C \times C^{\text{op}} \cong C^{\text{op}} \times C \xrightarrow{\text{Hom}_C} \text{Set} \]

For each \( C \)-morphism \( Y \xrightarrow{f} X \), the morphism \( yY \xrightarrow{yf} yX \) in \( \text{Set}^{\text{C}^{\text{op}}} \) is the natural transformation whose component at any given \( Z \in C^{\text{op}} \) is the function

\[
\begin{array}{ccc}
yY(Z) & \xrightarrow{(yf)_Z} & yX(Z) \\
\| & \| & \\
C(Z,Y) & \xrightarrow{g} & C(Z,X)
\end{array}
\]

\[ \xrightarrow{f \circ g} \]
Yoneda functor

\[ y : C \to \text{Set}_{\text{C}^{\text{op}}} \]

(where \( \text{C} \) is a small category)

is the Curried version of the \( \text{hom} \) functor

\[ \text{C} \times \text{C}^{\text{op}} \cong \text{C}^{\text{op}} \times \text{C} \xrightarrow{\text{Hom}_\text{C}} \text{Set} \]

For each \( \text{C} \)-morphism \( Y \stackrel{f}{\to} X \), the morphism \( yY \to yX \) in \( \text{Set}_{\text{C}^{\text{op}}} \) is the natural transformation whose component at any given \( Z \in \text{C}^{\text{op}} \) is the function

\[ yY(Z) \xrightarrow{(yf)_Z} yX(Z) \]

\[ \text{C}(Z,Y) \xrightarrow{f^*} \text{C}(Z,X) \]

this function is often written as \( f_* \)
The Yoneda Lemma

For each small category $\mathbf{C}$, each object $X \in \mathbf{C}$ and each presheaf $F \in \mathbf{Set}^{\mathbf{C}^{\text{op}}}$, there is a bijection of sets

$$\eta_{X,F} : \mathbf{Set}^{\mathbf{C}^{\text{op}}}(yX, F) \cong F(X)$$

which is natural in both $X$ and $F$. 

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The Yoneda Lemma

For each small category \( \mathbf{C} \), each object \( X \in \mathbf{C} \) and each presheaf \( F \in \text{Set}^{\mathbf{C}^{\text{op}}} \), there is a bijection of sets

\[
\eta_{X,F} : \text{Set}^{\mathbf{C}^{\text{op}}} (yX, F) \cong F(X)
\]

which is natural in both \( X \) and \( F \).
The Yoneda Lemma

For each small category $\mathbf{C}$, each object $X \in \mathbf{C}$ and each presheaf $F \in \textbf{Set}^{\mathbf{C}^{\text{op}}}$, there is a bijection of sets

$$\eta_{X,F} : \textbf{Set}^{\mathbf{C}^{\text{op}}}(yX, F) \cong F(X)$$

which is natural in both $X$ and $F$.

Definition of the function $\eta_{X,F} : \textbf{Set}^{\mathbf{C}^{\text{op}}}(yX, F) \rightarrow F(X)$:

for each $\theta : yX \rightarrow F$ in $\textbf{Set}^{\mathbf{C}^{\text{op}}}$ we have the function

$\mathbf{C}(X, X) = yX(X) \xrightarrow{\theta_X} F(X)$

and define

$$\eta_{X,F}(\theta) \triangleq \theta_X(id_X)$$
The Yoneda Lemma

For each small category $\mathbf{C}$, each object $X \in \mathbf{C}$ and each presheaf $F \in \mathbf{Set}^{\mathbf{C}^{\text{op}}}$, there is a bijection of sets

$$\eta_{X,F} : \mathbf{Set}^{\mathbf{C}^{\text{op}}}(yX, F) \cong F(X)$$

which is natural in both $X$ and $F$.

Definition of the function $\eta_{X,F}^{-1} : F(X) \to \mathbf{Set}^{\mathbf{C}^{\text{op}}}(yX, F)$:

for each $x \in X$, $Y \in \mathbf{C}$ and $f \in yX(Y) = \mathbf{C}(Y, X)$,

we get a $F(X) \xrightarrow{F(f)} F(Y)$ in $\mathbf{Set}$ and hence $F(f)(x) \in F(Y)$;
The Yoneda Lemma

For each small category $\mathcal{C}$, each object $X \in \mathcal{C}$ and each presheaf $F \in \text{Set}^{\mathcal{C}^{\text{op}}}$, there is a bijection of sets

$$\eta_{X,F} : \text{Set}^{\mathcal{C}^{\text{op}}} (yX, F) \cong F(X)$$

which is natural in both $X$ and $F$.

**Definition of the function** $\eta_{X,F}^{-1} : F(X) \to \text{Set}^{\mathcal{C}^{\text{op}}} (yX, F)$:

for each $x \in X$, $Y \in \mathcal{C}$ and $f \in yX(Y) = \mathcal{C}(Y, X)$,
we get a $F(X) \xrightarrow{F(f)} F(Y)$ in $\text{Set}$ and hence $F(f)(x) \in F(Y)$;
Define $\left( \eta_{X,F}^{-1}(x) \right)_Y : yX(Y) \to F(Y)$ by

$$\left( \eta_{X,F}^{-1}(x) \right)_Y (f) \triangleq F(f)(x)$$

check this gives a natural transformation $\eta_{X,F}^{-1}(x) : yX \to F$
Proof of \[ \eta_{X,F} \circ \eta_{X,F}^{-1} = \text{id}_{F(X)} \]

For any \( x \in F(X) \) we have

\[
\eta_{X,F} \left( \eta_{X,F}^{-1}(x) \right) \triangleq \left( \eta_{X,F}^{-1}(x) \right)_X (\text{id}_X) \\
\triangleq F(\text{id}_X)(x) \\
= \text{id}_{F(X)}(x) \\
= x
\]

by definition of \( \eta_{X,F} \)

by definition of \( \eta_{X,F}^{-1} \)

since \( F \) is a functor
Proof of \[ \eta_{X,F}^{-1} \circ \eta_{X,F} = \text{id}_{\text{Set}^{\text{op}}(yX,F)} \]

For any \( yX \overset{\theta}{\rightarrow} F \) in \( \text{Set}^{\text{op}} \) and \( Y \overset{f}{\rightarrow} X \) in \( \mathbf{C} \), we have

\[
\left( \eta_{X,F}^{-1}(\eta_{X,F}(\theta)) \right)_Y f \triangleq \left( \eta_{X,F}^{-1}(\theta_X(id_X)) \right)_Y f \\
\triangleq F(f)(\theta_X(id_X)) \\
= \theta_Y(f^*(id_X)) \\
\triangleq \theta_Y(id_X \circ f) \\
= \theta_Y(f)
\]

by definition of \( \eta_{X,F} \)
by definition of \( \eta_{X,F}^{-1} \)
by naturality of \( \theta \)
by definition of \( f^* \)
Proof of $\eta_{X,F}^{-1} \circ \eta_{X,F} = \text{id}_{\text{Set}^{\text{op}}(yX,F)}$

For any $yX \xrightarrow{\theta} F$ in $\text{Set}^{\text{op}}$ and $Y \xrightarrow{f} X$ in $C$, we have

$$\left(\eta_{X,F}^{-1}(\eta_{X,F}(\theta))\right)_Y f \triangleq \left(\eta_{X,F}^{-1}(\theta_X(id_X)))\right)_Y f$$

by definition of $\eta_{X,F}$

$$\triangleq F(f)(\theta_X(id_X))$$

by definition of $\eta_{X,F}^{-1}$

$$= \theta_Y(f^*(id_X))$$

by naturality of $\theta$

$$\triangleq \theta_Y(id_X \circ f)$$

by definition of $f^*$

$$= \theta_Y(f)$$

so $\forall \theta, Y, \left(\eta_{X,F}^{-1}(\eta_{X,F}(\theta))\right)_Y = \theta_Y$

so $\forall \theta, \eta_{X,F}^{-1}(\eta_{X,F}(\theta)) = \theta$

so $\eta_{X,F}^{-1} \circ \eta_{X,F} = \text{id}$. 
The Yoneda Lemma

For each small category \( \mathcal{C} \), each object \( X \in \mathcal{C} \) and each presheaf \( F \in \text{Set}^{\mathcal{C}^{\text{op}}} \), there is a bijection of sets

\[
\eta_{X,F} : \text{Set}^{\mathcal{C}^{\text{op}}} (yX, F) \cong F(X)
\]

which is natural in both \( X \) and \( F \).
Proof that $\eta_{X,F}$ is natural in $F$:

Given $F \xrightarrow{\varphi} G$ in $\text{Set}^{\text{op}}$, have to show that

$$\text{Set}^{\text{op}}(yX, F) \xrightarrow{\eta_{X,F}} F(X)$$

$$\xrightarrow{\varphi_*}$$

$$\text{Set}^{\text{op}}(yX, G) \xrightarrow{\eta_{X,G}} G(X)$$

commutes in $\text{Set}$. For all $yX \xrightarrow{\theta} F$ we have

$$\varphi_X(\eta_{X,F}(\theta)) \triangleq \varphi_X(\theta_X(id_X))$$

$$\triangleq (\varphi \circ \theta)_X(id_X)$$

$$\triangleq \eta_{X,G}(\varphi \circ \theta)$$

$$\triangleq \eta_{X,G}(\varphi_*(\theta))$$
Proof that $\eta_{X,F}$ is natural in $X$:

Given $Y \xrightarrow{f} X$ in $C$, have to show that

$$
\begin{align*}
\text{Set}^{\text{op}}(yX, F) & \xrightarrow{\eta_{X,F}} F(X) \\
(yf)^* & \downarrow \\
\text{Set}^{\text{op}}(yY, F) & \xrightarrow{\eta_{Y,F}} F(Y)
\end{align*}
$$

commutes in $\text{Set}$. For all $yX \xrightarrow{\theta} F$ we have

$$
F(f)((\eta_{X,F}(\theta))) \triangleq F(f)(\theta_X(\text{id}_X))
= \theta_Y(f^*(\text{id}_X))
= \theta_Y(f)
= \theta_Y(f_*(\text{id}_Y))
\triangleq (\theta \circ yf)_Y(\text{id}_Y)
\triangleq \eta_{Y,F}(\theta \circ yf)
\triangleq \eta_{Y,F}((yf)^*(\theta))
$$

by naturality of $\theta$
Corollary of the Yoneda Lemma:

the functor $y : C \to \text{Set}^{\text{C}^{\text{op}}}$ is full and faithful.

In general, a functor $F : C \to D$ is

- **faithful** if for all $X, Y \in C$ the function

  $\text{C}(X, Y) \to \text{D}(F(X), F(Y))$

  $f \mapsto F(f)$

  is injective:

  $\forall f, f' \in \text{C}(X, Y), \ F(f) = F(f') \Rightarrow f = f'$

- **full** if the above functions are all surjective:

  $\forall g \in \text{D}(F(X), F(Y)), \exists f \in \text{C}(X, Y), \ F(f) = g$