Lecture 15

Presheaf categories

Let $\mathbf{C}$ be a small category. The functor category $\boxed{\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}}$ is called the category of preseaves on $\mathbf{C}$.

- objects are contravariant functors from $\mathbf{C}$ to $\mathbf{Set}$
- morphisms are natural transformations

Much used in the semantics of various dependently-typed languages and logics.

Yoneda functor

$$\boxed{\mathbf{y} : \mathbf{C} \to \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}}$$

(where $\mathbf{C}$ is a small category)

is the Curried version of the hom functor

$$\mathbf{C} \times \mathbf{C}^{\mathrm{op}} \cong \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \xrightarrow{\mathtt{Hom}_{\mathbf{C}}} \mathbf{Set}$$

Yoneda functor

$$\boxed{\mathbf{y} : \mathbf{C} \to \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}}$$

(where $\mathbf{C}$ is a small category)

is the Curried version of the hom functor

$$\mathbf{C} \times \mathbf{C}^{\mathrm{op}} \cong \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \xrightarrow{\mathtt{Hom}_{\mathbf{C}}} \mathbf{Set}$$

- For each $\mathbf{C}$-object X, the object $\mathbf{y}X \in \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$ is the functor $\mathbf{C}(\_, X) : \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$ given by

$$\begin{array}{ccccc}
Z & \mapsto & \mathbf{C}(Z, X) & & g \circ f \\
\big\downarrow f & \mapsto & \big\uparrow & & \big\uparrow \\
Y & \mapsto & \mathbf{C}(Y, X) & & g
\end{array}$$

Yoneda functor

$$\boxed{\mathbf{y} : \mathbf{C} \to \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}}$$

(where $\mathbf{C}$ is a small category)

is the Curried version of the hom functor

$$\mathbf{C} \times \mathbf{C}^{\mathrm{op}} \cong \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \xrightarrow{\mathrm{Hom}_{\mathbf{C}}} \mathbf{Set}$$

- For each $\mathbf{C}$-object X, the object $\mathbf{y}X \in \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$ is the functor $\mathbf{C}(\_, X) : \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$ given by

$$\begin{array}{ccccc} Z & \mapsto & \mathbf{C}(Z, X) & & g \circ f \\ \downarrow f & \mapsto & \uparrow & & \uparrow f^* \\ Y & \mapsto & \mathbf{C}(Y, X) & & g \end{array}$$

this function is often written as f^*

Yoneda functor

$$\boxed{\mathbf{y} : \mathbf{C} \to \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}}$$

(where $\mathbf{C}$ is a small category)

is the Curried version of the hom functor

$$\mathbf{C} \times \mathbf{C}^{\mathrm{op}} \cong \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \xrightarrow{\mathrm{Hom}_{\mathbf{C}}} \mathbf{Set}$$

- For each $\mathbf{C}$-morphism $Y \xrightarrow{f} X$, the morphism $\mathbf{y}Y \xrightarrow{\mathbf{y}f} \mathbf{y}X$ in $\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$ is the natural transformation whose component at any given $Z \in \mathbf{C}^{\mathrm{op}}$ is the function

$$\begin{array}{ccc} \mathbf{y}Y(Z) & \xrightarrow{(\mathbf{y}f)_Z} & \mathbf{y}X(Z) \\ \| & & \| \\ \mathbf{C}(Z,Y) & & \mathbf{C}(Z,X) \\ & & \\ g & \longmapsto & f \circ g \end{array}$$

Yoneda functor

$$\boxed{\mathbf{y} : \mathbf{C} \to \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}}$$

(where $\mathbf{C}$ is a small category)

is the Curried version of the hom functor

$$\mathbf{C} \times \mathbf{C}^{\mathrm{op}} \cong \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \xrightarrow{\mathrm{Hom}_{\mathbf{C}}} \mathbf{Set}$$

- For each $\mathbf{C}$-morphism $Y \xrightarrow{f} X$, the morphism $\mathbf{y}Y \xrightarrow{\mathbf{y}f} \mathbf{y}X$ in $\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$ is the natural transformation whose component at any given $Z \in \mathbf{C}^{\mathrm{op}}$ is the function

$$\begin{array}{ccc} \mathbf{y}Y(Z) & \xrightarrow{(\mathbf{y}f)_Z} & \mathbf{y}X(Z) \\ \| & & \| \\ \mathbf{C}(Z,Y) & & \mathbf{C}(Z,X) \\ & f_* & \\ g & \longmapsto & f \circ g \end{array}$$

this function is often written as f_*

The Yoneda Lemma

For each small category $\mathbf{C}$, each object $X \in \mathbf{C}$ and each presheaf $F \in \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$, there is a bijection of sets

$$\eta_{X,F} : \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}(\mathrm{y}X, F) \cong F(X)$$

which is natural in both X and F.

The Yoneda Lemma

For each small category $\mathbf{C}$, each object $X \in \mathbf{C}$ and each presheaf $F \in \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$, there is a bijection of sets

$$\eta_{X,F} : \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}(\mathrm{y}X, F) \cong F(X)$$

which is natural in both X and F.

the value of
$F : \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$
at X

the set of natural transformations from
the functor $\mathrm{y}X : \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$
to the functor $F : \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$

The Yoneda Lemma

For each small category $\mathbf{C}$, each object $X \in \mathbf{C}$ and each presheaf $F \in \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$, there is a bijection of sets

$$\eta_{X,F} : \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}(\mathrm{y}X, F) \cong F(X)$$

which is natural in both X and F.

Definition of the function $\eta_{X,F} : \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}(\mathrm{y}X, F) \to F(X)$:

for each $\theta : \mathrm{y}X \to F$ in $\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$ we have the function $\mathbf{C}(X, X) = \mathrm{y}X(X) \xrightarrow{\theta_X} F(X)$ and define

$$\boxed{\eta_{X,F}(\theta) \triangleq \theta_X(\mathtt{id}_X)}$$

The Yoneda Lemma

For each small category $\mathbf{C}$, each object $X \in \mathbf{C}$ and each presheaf $F \in \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$, there is a bijection of sets

$$\eta_{X,F} : \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}(\mathrm{y}X, F) \cong F(X)$$

which is natural in both X and F.

Definition of the function $\eta_{X,F}^{-1} : F(X) \to \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}(\mathrm{y}X, F)$:

for each $x \in X$, $Y \in \mathbf{C}$ and $f \in \mathrm{y}X(Y) = \mathbf{C}(Y, X)$,

we get a $F(X) \xrightarrow{F(f)} F(Y)$ in $\mathbf{Set}$ and hence $F(f)(x) \in F(Y)$;

The Yoneda Lemma

For each small category $\mathbf{C}$, each object $X \in \mathbf{C}$ and each presheaf $F \in \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$, there is a bijection of sets

$$\eta_{X,F} : \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}(\mathrm{y}X, F) \cong F(X)$$

which is natural in both X and F.

Definition of the function $\eta^{-1}_{X,F} : F(X) \to \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}(\mathrm{y}X, F)$:

for each $x \in X$, $Y \in \mathbf{C}$ and $f \in \mathrm{y}X(Y) = \mathbf{C}(Y, X)$,

we get a $F(X) \xrightarrow{F(f)} F(Y)$ in $\mathbf{Set}$ and hence $F(f)(x) \in F(Y)$;

Define $\left(\eta^{-1}_{X,F}(x)\right)_Y : \mathrm{y}X(Y) \to F(Y)$ by

$$\boxed{\left(\eta^{-1}_{X,F}(x)\right)_Y (f) \triangleq F(f)(x)}$$

check this gives a natural transformation $\eta^{-1}_{X,F}(x) : \mathrm{y}X \to F$

Proof of $\boxed{\eta_{X,F} \circ \eta^{-1}_{X,F} = \mathtt{id}_{F(X)}}$

For any $x \in F(X)$ we have

$$\begin{aligned}
\eta_{X,F}\left(\eta^{-1}_{X,F}(x)\right) &\triangleq \left(\eta^{-1}_{X,F}(x)\right)_X(\mathtt{id}_X) && \text{by definition of } \eta_{X,F} \\
&\triangleq F(\mathtt{id}_X)(x) && \text{by definition of } \eta^{-1}_{X,F} \\
&= \mathtt{id}_{F(X)}(x) && \text{since } F \text{ is a functor} \\
&= x
\end{aligned}$$

Proof of $\boxed{\eta^{-1}_{X,F} \circ \eta_{X,F} = \mathtt{id}_{\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}(\mathbf{y}X,F)}}$

For any $\mathbf{y}X \xrightarrow{\theta} F$ in $\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$ and $Y \xrightarrow{f} X$ in $\mathbf{C}$, we have

$$
\begin{aligned}
\left(\eta^{-1}_{X,F}\,(\eta_{X,F}(\theta))\right)_Y f &\triangleq \left(\eta^{-1}_{X,F}\,(\theta_X(\mathtt{id}_X))\right)_Y f && \text{by definition of } \eta_{X,F} \\
&\triangleq F(f)(\theta_X(\mathtt{id}_X)) && \text{by definition of } \eta^{-1}_{X,F} \\
&= \theta_Y(f^*(id_X)) && \text{by naturality of } \theta \\
&\triangleq \theta_Y(\mathtt{id}_X \circ f) && \text{by definition of } f^* \\
&= \theta_Y(f)
\end{aligned}
$$

naturality of θ

$$
\begin{tikzcd}
\mathbf{y}X(Y) \arrow[r, "\theta_Y"] & F(Y) \\
\mathbf{y}X(X) \arrow[u, "f^*"] \arrow[r, "\theta_X"'] & F(X) \arrow[u, "F(f)"']
\end{tikzcd}
$$

Proof of $\boxed{\eta^{-1}_{X,F} \circ \eta_{X,F} = \mathtt{id}_{\mathbf{Set}^{\mathbf{C}^{op}}(\mathbf{y}X,F)}}$

For any $\mathbf{y}X \xrightarrow{\theta} F$ in $\mathbf{Set}^{\mathbf{C}^{op}}$ and $Y \xrightarrow{f} X$ in $\mathbf{C}$, we have

$$\begin{aligned}
\left(\eta^{-1}_{X,F}(\eta_{X,F}(\theta))\right)_Y f &\triangleq \left(\eta^{-1}_{X,F}(\theta_X(\mathtt{id}_X))\right)_Y f && \text{by definition of } \eta_{X,F} \\
&\triangleq F(f)(\theta_X(\mathtt{id}_X)) && \text{by definition of } \eta^{-1}_{X,F} \\
&= \theta_Y(f^*(id_X)) && \text{by naturality of } \theta \\
&\triangleq \theta_Y(\mathtt{id}_X \circ f) && \text{by definition of } f^* \\
&= \theta_Y(f)
\end{aligned}$$

so $\forall \theta, Y,\ \left(\eta^{-1}_{X,F}(\eta_{X,F}(\theta))\right)_Y = \theta_Y$

so $\forall \theta,\ \eta^{-1}_{X,F}(\eta_{X,F}(\theta)) = \theta$

so $\eta^{-1}_{X,F} \circ \eta_{X,F} = \mathtt{id}$.

The Yoneda Lemma

For each small category $\mathbf{C}$, each object $X \in \mathbf{C}$ and each presheaf $F \in \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$, there is a bijection of sets

$$\eta_{X,F} : \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}(\mathrm{y}X, F) \cong F(X)$$

which is natural in both X and F.

Proof that $\eta_{X,F}$ is natural in F:

Given $F \xrightarrow{\varphi} G$ in $\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$, have to show that

$$\begin{tikzcd}
\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}(\mathbf{y}X, F) \arrow[r, "\eta_{X,F}"] \arrow[d, "\varphi_*"'] & F(X) \arrow[d, "\varphi_X"] \\
\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}(\mathbf{y}X, G) \arrow[r, "\eta_{X,G}"'] & G(X)
\end{tikzcd}$$

commutes in $\mathbf{Set}$. For all $\mathbf{y}X \xrightarrow{\theta} F$ we have

$$\begin{aligned}
\varphi_X\,(\eta_{X,F}(\theta)) &\triangleq \varphi_X\,(\theta_X(\mathtt{id}_X)) \\
&\triangleq (\varphi \circ \theta)_X(\mathtt{id}_X) \\
&\triangleq \eta_{X,G}(\varphi \circ \theta) \\
&\triangleq \eta_{X,G}(\varphi_*(\theta))
\end{aligned}$$

Proof that $\eta_{X,F}$ is natural in X:

Given $Y \xrightarrow{f} X$ in $\mathbf{C}$, have to show that

$$\begin{tikzcd}
\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}(\mathrm{y}X, F) \arrow[r, "\eta_{X,F}"] \arrow[d, "(\mathrm{y}f)^*"'] & F(X) \arrow[d, "F(f)"] \\
\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}(\mathrm{y}Y, F) \arrow[r, "\eta_{Y,F}"'] & F(Y)
\end{tikzcd}$$

commutes in $\mathbf{Set}$. For all $\mathrm{y}X \xrightarrow{\theta} F$ we have

$$\begin{aligned}
F(f)((\eta_{X,F}(\theta)) &\triangleq F(f)(\theta_X(\mathtt{id}_X)) \\
&= \theta_Y(f^*(\mathtt{id}_X)) && \text{by naturality of } \theta \\
&= \theta_Y(f) \\
&= \theta_Y(f_*(\mathtt{id}_Y)) \\
&\triangleq (\theta \circ \mathrm{y}f)_Y(\mathtt{id}_Y) \\
&\triangleq \eta_{Y,F}(\theta \circ \mathrm{y}f) \\
&\triangleq \eta_{Y,F}((\mathrm{y}f)^*(\theta))
\end{aligned}$$

Corollary of the Yoneda Lemma:

the functor $\mathbf{y} : \mathbf{C} \to \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$ is full and faithful.

In general, a functor $F : \mathbf{C} \to \mathbf{D}$ is

- faithful if for all $X, Y \in \mathbf{C}$ the function

$$\begin{array}{ccc} \mathbf{C}(X,Y) & \to & \mathbf{D}(F(X),F(Y)) \\ f & \mapsto & F(f) \end{array}$$

  is injective:

$$\forall f, f' \in \mathbf{C}(X,Y),\ F(f) = F(f') \Rightarrow f = f'$$

- full if the above functions are all surjective:

$$\forall g \in \mathbf{D}(F(X),F(Y)), \exists f \in \mathbf{C}(X,Y),\ F(f) = g$$