

Lecture 13

Recall:

Given categories and functors $\mathbf{C} \begin{array}{c} \xrightarrow{F} \\ \xleftarrow{G} \end{array} \mathbf{D}$,

an adjunction $F \dashv G$ is specified by functions

$$\theta_{X,Y} \downarrow \frac{FX \xrightarrow{g} Y}{X \xrightarrow{\bar{g}} GY} \quad \uparrow \theta_{X,Y}^{-1} \frac{FX \xrightarrow{\bar{f}} Y}{X \xrightarrow{f} GY}$$

(for each $X \in \mathbf{C}$ and $Y \in \mathbf{D}$) satisfying $\bar{\bar{f}} = f$, $\bar{\bar{g}} = g$ and

$$\frac{FX' \xrightarrow{Fu} FX \xrightarrow{g} Y}{X' \xrightarrow{u} X \xrightarrow{\bar{g}} GY} \quad \frac{FX \xrightarrow{g} Y \xrightarrow{v} Y'}{X \xrightarrow{\bar{g}} GY \xrightarrow{Gv} GY'}$$

Theorem. A category \mathbf{C} has binary products iff the diagonal functor $\Delta = \langle \text{id}_{\mathbf{C}}, \text{id}_{\mathbf{C}} \rangle : \mathbf{C} \rightarrow \mathbf{C} \times \mathbf{C}$ has a right adjoint.

Theorem. A category \mathbf{C} with binary products also has all exponentials of pairs of objects iff for all $X \in \mathbf{C}$, the functor $(_) \times X : \mathbf{C} \rightarrow \mathbf{C}$ has a right adjoint.

Both these theorems are instances of the following theorem, a very useful characterisation of when a functor has a right adjoint (or dually, a left adjoint).

Characterisation of right adjoints

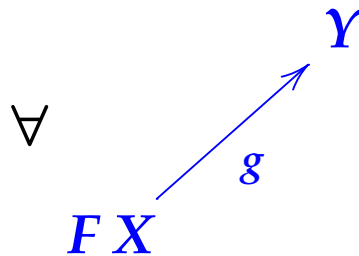
Theorem. A functor $F : \mathbf{C} \rightarrow \mathbf{D}$ has a right adjoint iff for all \mathbf{D} -objects $Y \in \mathbf{D}$, there is a \mathbf{C} -object $G Y \in \mathbf{C}$ and a \mathbf{C} -morphism $\varepsilon_Y : F(G Y) \rightarrow Y$ with the following “universal property”:

(UP) for all $X \in \mathbf{C}$ and $g \in \mathbf{D}(F X, Y)$
there is a unique $\bar{g} \in \mathbf{C}(X, G Y)$
satisfying $\varepsilon_Y \circ F(\bar{g}) = g$

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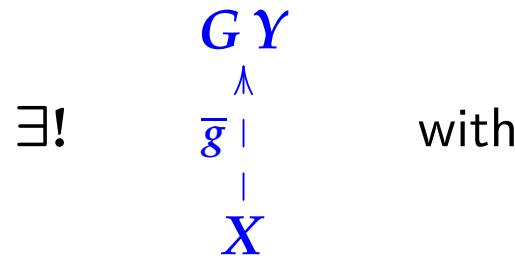
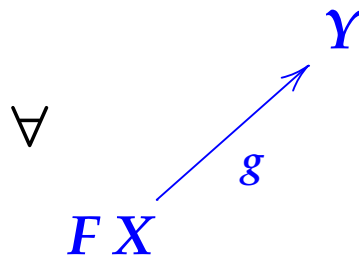
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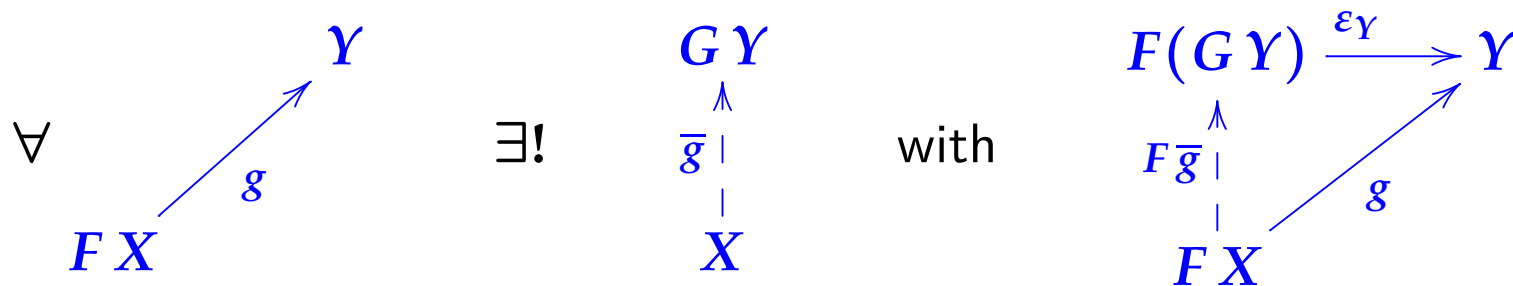
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Proof of the Theorem—“only if” part:

Given an adjunction (F, G, θ) , for each $Y \in \mathbf{D}$ we produce $\varepsilon_Y : F(GY) \rightarrow Y$ in \mathbf{D} satisfying (UP).

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We are given $\theta_{X,Y} : \mathbf{D}(FX, Y) \cong \mathbf{C}(X, GY)$, natural in X and Y . Define

$$\varepsilon_Y \triangleq \theta_{GY, Y}^{-1}(\text{id}_{GY}) : F(GY) \rightarrow Y$$

In other words $\varepsilon_Y = \overline{\text{id}_{GY}}$.

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Given any $\begin{cases} g : FX \rightarrow Y & \text{in } \mathbf{D} \\ f : X \rightarrow GY & \text{in } \mathbf{C} \end{cases}$, by naturality of θ we have

$$\frac{FX \xrightarrow{g} Y}{X \xrightarrow{\bar{g}} GY} \text{ and } \frac{\varepsilon_Y \circ Ff : FX \xrightarrow{Ff} F(GY) \xrightarrow{\overline{\text{id}_{GY}}} Y}{f : X \xrightarrow{f} GY \xrightarrow{\text{id}_{GY}} GY}$$

Hence $g = \varepsilon_Y \circ F\bar{g}$ and $g = \varepsilon_Y \circ Ff \Rightarrow \bar{g} = f$.

Thus we do indeed have (UP).

Proof of the Theorem—“if” part:

We are given $F : \mathbf{C} \rightarrow \mathbf{D}$ and for each $Y \in \mathbf{D}$ a \mathbf{C} -object GY and \mathbf{C} -morphism $\varepsilon_Y : F(GY) \rightarrow Y$ satisfying (UP). We have to

1. extend $Y \mapsto GY$ to a functor $G : \mathbf{D} \rightarrow \mathbf{C}$
2. construct a natural isomorphism
$$\theta : \text{Hom}_{\mathbf{D}} \circ (F^{\text{op}} \times \text{id}_{\mathbf{D}}) \cong \text{Hom}_{\mathbf{C}} \circ (\text{id}_{\mathbf{C}^{\text{op}}} \times G)$$

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For each \mathbf{D} -morphism $g : Y' \rightarrow Y$ we get $F(GY') \xrightarrow{\varepsilon_{Y'}} Y' \xrightarrow{g} Y$ and can apply (UP) to get

$$Gg \triangleq \overline{g \circ \varepsilon_{Y'}} : GY' \rightarrow GY$$

The uniqueness part of (UP) implies

$$G \text{id} = \text{id} \quad \text{and} \quad G(g' \circ g) = Gg' \circ Gg$$

so that we get a functor $G : \mathbf{D} \rightarrow \mathbf{C}$. \square

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Since for all $g : FX \rightarrow Y$ there is a unique $f : X \rightarrow GY$ with $g = \varepsilon_Y \circ Ff$,

$$f \mapsto \bar{f} \triangleq \varepsilon_Y \circ Ff$$

determines a bijection $\mathbf{C}(X, GY) \cong \mathbf{C}(FX, Y)$; and it is natural in X & Y because

$$\begin{aligned} \overline{Gv \circ f \circ u} &\triangleq \varepsilon_{Y'} \circ F(Gv \circ f \circ u) \\ &= (\varepsilon_{Y'} \circ F(Gv)) \circ Ff \circ Fu && \text{since } F \text{ is a functor} \\ &= (v \circ \varepsilon_Y) \circ Ff \circ Fu && \text{by definition of } Gv \\ &= v \circ \bar{f} \circ Fu && \text{by definition of } \bar{f} \end{aligned}$$

So we can take θ to be the inverse of this natural isomorphism. \square

Dual of the Theorem:

$G : \mathbf{C} \leftarrow \mathbf{D}$ has a **left** adjoint iff for all $X \in \mathbf{C}$ there are $F X \in \mathbf{D}$ and $\eta_X \in \mathbf{C}(X, G(F X))$ with the universal property:

for all $Y \in \mathbf{D}$ and $f \in \mathbf{C}(X, G Y)$
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E.g. we can conclude that **the forgetful functor** $U : \mathbf{Mon} \rightarrow \mathbf{Set}$ has a **left adjoint** $F : \mathbf{Set} \rightarrow \mathbf{Mon}$, because of the universal property of

$$F \Sigma \triangleq (\text{List } \Sigma, @, \text{nil}) \quad \text{and} \quad \eta_\Sigma : \Sigma \rightarrow \text{List } \Sigma$$

noted in Lecture 3.

Why are adjoint functors important/useful?

Their universal property (UP) usually embodies some useful mathematical construction

(e.g. “freely generated structures are left adjoints for forgetting-structure”)

and pins it down uniquely up to isomorphism.