Lecture 13

Recall:

Given categories and functors $\mathbf{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathbf{D}$,

an adjunction $\boxed{F \dashv G}$ is specified by functions

$$\theta_{X,Y}\downarrow \frac{F\,X \xrightarrow{g} Y}{X \xrightarrow{\overline{g}} G\,Y} \qquad \uparrow\theta^{-1}_{X,Y} \frac{F\,X \xrightarrow{\overline{f}} Y}{X \xrightarrow{f} G\,Y}$$

(for each $X \in \mathbf{C}$ and $Y \in \mathbf{D}$) satisfying $\overline{\overline{f}} = f$, $\overline{\overline{g}} = g$ and

$$\frac{F\,X' \xrightarrow{F\,u} F\,X \xrightarrow{g} Y}{X' \xrightarrow{u} X \xrightarrow{\overline{g}} G\,Y} \qquad \frac{F\,X \xrightarrow{g} Y \xrightarrow{v} Y'}{X \xrightarrow{\overline{g}} G\,Y \xrightarrow{G\,v} G\,Y'}$$

Theorem. A category $\mathbf{C}$ has binary products iff the diagonal functor $\boldsymbol{\Delta} = \langle \mathtt{id}_{\mathbf{C}}, \mathtt{id}_{C} \rangle : \mathbf{C} \to \mathbf{C} \times \mathbf{C}$ has a right adjoint.

Theorem. A category $\mathbf{C}$ with binary products also has all exponentials of pairs of objects iff for all $X \in \mathbf{C}$, the functor $(\_) \times X : \mathbf{C} \to \mathbf{C}$ has a right adjoint.

Both these theorems are instances of the following theorem, a very useful characterisation of when a functor has a right adjoint (or dually, a left adjoint).

Characterisation of right adjoints

Theorem. A functor $F : \mathbf{C} \to \mathbf{D}$ has a right adjoint iff for all $\mathbf{D}$-objects $Y \in \mathbf{D}$, there is a $\mathbf{C}$-object $G\,Y \in \mathbf{C}$ and a $\mathbf{C}$-morphism $\varepsilon_Y : F(G\,Y) \to Y$ with the following "universal property":

(UP) $\boxed{\begin{array}{l}\text{for all } X \in \mathbf{C} \text{ and } g \in \mathbf{D}(F\,X, Y) \\ \text{there is a unique } \overline{g} \in \mathbf{C}(X, G\,Y) \\ \text{satisfying } \varepsilon_Y \circ F(\overline{g}) = g\end{array}}$

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with
$$\begin{tikzcd} F(G\,Y) \arrow[r, "\varepsilon_Y"] & Y \\ F\,X \arrow[u, dashed, "F\,\overline{g}"] \arrow[ur, "g"'] & \end{tikzcd}$$

Proof of the Theorem—“only if” part:

Given an adjunction (F, G, θ), for each $Y \in \mathbf{D}$ we produce $\varepsilon_Y : F(G\,Y) \to Y$ in $\mathbf{D}$ satisfying (UP).

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We are given $\theta_{X,Y} : \mathbf{D}(F\,X, Y) \cong \mathbf{C}(X, G\,Y)$, natural in X and Y. Define

$$\boxed{\varepsilon_Y \triangleq \theta^{-1}_{G\,Y,Y}(\mathtt{id}_{G\,Y}) : F(G\,Y) \to Y}$$

In other words $\varepsilon_Y = \overline{\mathtt{id}_{G\,Y}}$.

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Given any $\begin{cases} g : F\,X \to Y & \text{in } \mathbf{D} \\ f : X \to G\,Y & \text{in } \mathbf{C} \end{cases}$, by naturality of θ we have

$$\frac{F\,X \xrightarrow{g} Y}{X \xrightarrow{\overline{g}} G\,Y} \text{ and } \frac{\varepsilon_Y \circ F\,f : F\,X \xrightarrow{F\,f} F(G\,Y) \xrightarrow{\overline{\mathtt{id}_{G\,Y}}} Y}{f : X \xrightarrow{f} G\,Y \xrightarrow{\mathtt{id}_{G\,Y}} G\,Y}$$

Hence $g = \varepsilon_Y \circ F\,\overline{g}$ and $g = \varepsilon_Y \circ F\,f \;\Rightarrow\; \overline{g} = f$.

Thus we do indeed have (UP).

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We are given $F : \mathbf{C} \to \mathbf{D}$ and for each $Y \in \mathbf{D}$ a $\mathbf{C}$-object $G\,Y$ and $\mathbf{C}$-morphism $\varepsilon_Y : F(G\,Y) \to Y$ satisfying (UP). We have to

1. *extend* $Y \mapsto G\,Y$ *to a functor* $G : \mathbf{D} \to \mathbf{C}$
2. *construct a natural isomorphism*
   $\theta : \mathtt{Hom}_{\mathbf{D}} \circ (F^{\mathrm{op}} \times \mathtt{id}_{\mathbf{D}}) \cong \mathtt{Hom}_{\mathbf{C}} \circ (\mathtt{id}_{\mathbf{C}^{\mathrm{op}}} \times G)$

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For each $\mathbf{D}$-morphism $g : Y' \to Y$ we get $F(G\,Y') \xrightarrow{\varepsilon_{Y'}} Y' \xrightarrow{g} Y$ and can apply (UP) to get

$$G\,g \triangleq \overline{g \circ \varepsilon_{Y'}} : G\,Y' \to G\,Y$$

The uniqueness part of (UP) implies

$$G\,\mathtt{id} = \mathtt{id} \quad \text{and} \quad G(g' \circ g) = G\,g' \circ G\,g$$

so that we get a functor $G : \mathbf{D} \to \mathbf{C}$. $\square$

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Since for all $g : F\,X \to Y$ there is a unique $f : X \to G\,Y$ with $g = \varepsilon_Y \circ F\,f$,

$$f \mapsto \overline{f} \triangleq \varepsilon_Y \circ F\,f$$

determines a bijection $\mathbf{C}(X, G\,Y) \cong \mathbf{C}(F\,X, Y)$; and it is natural in X & Y because

$$\begin{aligned}
\overline{G\,v \circ f \circ u} &\triangleq \varepsilon_{Y'} \circ F(G\,v \circ f \circ u) \\
&= (\varepsilon_{Y'} \circ F(G\,v)) \circ F\,f \circ F\,u && \text{since } F \text{ is a functor} \\
&= (v \circ \varepsilon_Y) \circ F\,f \circ F\,u && \text{by definition of } G\,v \\
&= v \circ \overline{f} \circ F\,u && \text{by definition of } \overline{f}
\end{aligned}$$

So we can take θ to be the inverse of this natural isomorphism. $\square$

Dual of the Theorem:

$G : \mathbf{C} \leftarrow \mathbf{D}$ has a left adjoint iff for all $X \in \mathbf{C}$ there are $F\,X \in \mathbf{D}$ and $\eta_X \in \mathbf{C}(X, G(F\,X))$ with the universal property:

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E.g. we can conclude that the forgetful functor $U : \mathbf{Mon} \to \mathbf{Set}$ has a left adjoint $F : \mathbf{Set} \to \mathbf{Mon}$, because of the universal property of

$$F\,\Sigma \triangleq (\mathtt{List}\,\Sigma, @, \mathtt{nil}) \quad \text{and} \quad \eta_\Sigma : \Sigma \to \mathtt{List}\,\Sigma$$

noted in Lecture 3.

Why are adjoint functors important/useful?

Their universal property (UP) usually embodies some useful mathematical construction

(e.g. “freely generated structures are left adjoints for forgetting-stucture”)

and pins it down uniquely up to isomorphism.