Lecture 12

Adjoint functors

The concepts of “category”, “functor” and “natural transformation” were invented by Eilenberg and MacLane in order to formalise “adjoint situations”.

They appear everywhere in mathematics, logic and (hence) computer science.

Examples of adjoint situations that we have already seen. . .

Free monoids

$$\frac{\Sigma \to U(M, \cdot, e) \quad \text{morphisms in } \textbf{Set}}{F\Sigma \to (M, \cdot, e) \quad \text{morphisms in } \textbf{Mon}}$$

bijection

$$\textbf{Set}(\Sigma, M) \cong \textbf{Mon}(F\Sigma, (M, \cdot, e))$$

$$f \mapsto \hat{f}$$

$$g \circ \eta_\Sigma \leftarrow g$$

(where $\eta_\Sigma : \Sigma \to F\Sigma = \texttt{List}\,\Sigma$ is $a \mapsto [a]$)

The bijection is "natural in Σ and $(M, \cdot, e)$" (to be explained)

Binary product in a category $\mathbf{C}$

$$\frac{(Z,Z) \to (X,Y) \text{ morphisms in } \mathbf{C}\times\mathbf{C}}{Z \to X \times Y \text{ morphisms in } \mathbf{C}}$$

bijection

$$(\mathbf{C}\times\mathbf{C})((Z,Z),(X,Y)) \cong \mathbf{C}(Z, X\times Y)$$
$$(f,g) \mapsto \langle f, g\rangle$$
$$(\pi_1 \circ h, \pi_2 \circ h) \leftarrow\!\shortmid h$$

This bijection is "natural in X, Y, Z" (to be explained)

Exponentials in a category **C** with binary products

$$\frac{Z \times X \to Y \text{ morphisms in } \mathbf{C}}{Z \to Y^X \text{ morphisms in } \mathbf{C}}$$

bijection

$$\mathbf{C}(Z \times X, Y) \cong \mathbf{C}(Z, Y^X)$$

$$f \mapsto \mathtt{cur}\, f$$

$$\mathtt{app} \circ (g \times \mathtt{id}_X) \leftarrow g$$

The bijection is "natural in X, Y, Z" (to be explained)

Adjunction

Definition. An adjunction between two categories $\mathbf{C}$ and $\mathbf{D}$ is specified by:

- functors $\mathbf{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathbf{D}$
- for each $X \in \mathbf{C}$ and $Y \in \mathbf{D}$ a bijection
  $\theta_{X,Y} : \mathbf{D}(F\,X, Y) \cong \mathbf{C}(X, G\,Y)$
  which is natural in X and Y.

for all $\begin{cases} u : X' \to X \text{ in } \mathbf{C} \\ v : Y \to Y' \text{ in } \mathbf{D} \end{cases}$ and all $g : F\,X \to Y$ in $\mathbf{D}$

$$X' \xrightarrow{u} X \xrightarrow{\theta_{X,Y}(g)} G\,Y \xrightarrow{G\,v} G\,Y' = \theta_{X',Y'}\left(F\,X' \xrightarrow{F\,u} F\,X \xrightarrow{g} Y \xrightarrow{v} Y'\right)$$

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what has this to do with the concept of natural transformation between functors?

Hom functors

If $\mathbf{C}$ is a locally small category, then we get a functor

$$\mathtt{Hom}_{\mathbf{C}} : \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \longrightarrow \mathbf{Set}$$

with $\mathtt{Hom}_{C}(X, Y) \triangleq \mathbf{C}(X, Y)$ and

$$\mathtt{Hom}_{\mathbf{C}}\left((X, Y) \xrightarrow{(f,g)} (X', Y')\right) \triangleq \mathbf{C}(X, Y) \xrightarrow{\mathtt{Hom}_{C}(f,g)} \mathbf{C}(X', Y')$$

$$\mathtt{Hom}_{C}(f, g)\, h \triangleq g \circ h \circ f$$

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If $(f, g) : (X, Y) \to (X', Y')$ in $\mathbf{C}^{\mathrm{op}} \times \mathbf{C}$ and $h : X \to Y$ in $\mathbf{C}$,
then in $\mathbf{C}$ we have $f : X' \to X$, $g : Y \to Y'$ and so $g \circ h \circ f : X' \to Y'$

Natural isomorphisms

Given functors $F, G : \mathbf{C} \to \mathbf{D}$, a natural isomorphism $\theta : F \cong G$ is simply an isomorphism between F and G in the functor category $\mathbf{D}^{\mathbf{C}}$.

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Lemma. If $\theta : F \to G$ is a natural transformation and for each $X \in \mathbf{C}$, $\theta_X : F\,X \to G\,X$ is an isomorphism in $\mathbf{D}$, then the family of morphisms $(\theta_X^{-1} : G\,X \to F\,X \mid X \in \mathbf{C})$ gives a natural transformation $\theta^{-1} : G \to F$ which is inverse to θ in $\mathbf{D}^{\mathbf{C}}$ and hence θ is a natural isomorphism. □

An adjunction between locally small categories **C** and **D** is simply a triple (F, G, θ) where

- $\mathbf{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathbf{D}$
- θ is a natural isomorphism between the functors

$$\mathbf{C}^{\mathrm{op}} \times \mathbf{D} \xrightarrow{F^{\mathrm{op}} \times \mathrm{id}_{\mathbf{D}}} \mathbf{D}^{\mathrm{op}} \times \mathbf{D} \xrightarrow{\mathrm{Hom}_{\mathbf{D}}} \mathbf{Set}$$

and

$$\mathbf{C}^{\mathrm{op}} \times \mathbf{D} \xrightarrow{\mathrm{id}_{\mathbf{C}^{\mathrm{op}}} \times G} \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \xrightarrow{\mathrm{Hom}_{\mathbf{C}}} \mathbf{Set}$$

Terminology:

Given $\mathbf{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathbf{D}$

is there is some natural isomorphism

$\theta : \mathtt{Hom}_{\mathbf{D}} \circ (F^{\mathrm{op}} \times \mathtt{id}_{\mathbf{D}}) \cong \mathtt{Hom}_{\mathbf{C}} \circ (\mathtt{id}_{\mathbf{C}^{\mathrm{op}}} \times G)$

one says

F is a left adjoint for G

G is a right adjoint for F

and writes

$$\boxed{F \dashv G}$$

Notation associated with an adjunction (F, G, θ)

Given $\begin{cases} g : F\,X \to Y \\ f : X \to G\,Y \end{cases}$

we write $\begin{cases} \overline{g} & \triangleq \theta_{X,Y}(g) : X \to G\,Y \\ \overline{f} & \triangleq \theta_{X,Y}^{-1}(f) : F\,X \to Y \end{cases}$

Thus $\overline{\overline{g}} = g$, $\overline{\overline{f}} = f$ and naturality of $\theta_{X,Y}$ in X and Y means that

$$\overline{v \circ g \circ F\,u} = G\,v \circ \overline{g} \circ u$$

Notation associated with an adjunction (F, G, θ)

The existence of θ is sometimes indicated by writing

$$\frac{F\,X \xrightarrow{g} Y}{X \xrightarrow{\overline{g}} G\,Y}$$

Using this notation, one can split the naturality condition for θ into two:

$$\frac{F\,X' \xrightarrow{F\,u} F\,X \xrightarrow{g} Y}{X' \xrightarrow{u} X \xrightarrow{\overline{g}} G\,Y} \qquad \frac{F\,X \xrightarrow{g} Y \xrightarrow{v} Y'}{X \xrightarrow{\overline{g}} G\,Y \xrightarrow{G\,v} G\,Y'}$$

Theorem. A category $\mathbf{C}$ has binary products iff the diagonal functor $\boldsymbol{\Delta} = \langle \mathtt{id}_{\mathbf{C}}, \mathtt{id}_{C} \rangle : \mathbf{C} \to \mathbf{C} \times \mathbf{C}$ has a right adjoint.

Theorem. A category $\mathbf{C}$ with binary products also has all exponentials of pairs of objects iff for all $X \in \mathbf{C}$, the functor $(\_) \times X : \mathbf{C} \to \mathbf{C}$ has a right adjoint.

Both these theorems are instances of the following theorem, a very useful characterisation of when a functor has a right adjoint (or dually, a left adjoint).

Characterisation of right adjoints

Theorem. A functor $F : \mathbf{C} \to \mathbf{D}$ has a right adjoint iff for all $\mathbf{D}$-objects $Y \in \mathbf{D}$, there is a $\mathbf{C}$-object $G\,Y \in \mathbf{C}$ and a $\mathbf{C}$-morphism $\varepsilon_Y : F(G\,Y) \to Y$ with the following "universal property":

(UP) for all $X \in \mathbf{C}$ and $g \in \mathbf{D}(F\,X, Y)$ there is a unique $\overline{g} \in \mathbf{C}(X, G\,Y)$ satisfying $\varepsilon_Y \circ F(\overline{g}) = g$