Lecture 11

The category of small categories

Recall definition of $\mathbf{Cat}$:

- objects are all small categories
- morphisms in $\mathbf{Cat}(\mathbf{C}, \mathbf{D})$ are all functors $\mathbf{C} \to \mathbf{D}$
- composition and identity morphisms as for functors

Cat has a terminal object

The category

$$\boxed{\begin{array}{c} \mathbf{0} \circlearrowleft \mathtt{id}_{\mathbf{0}} \\ \text{one object, one morphism} \end{array}}$$

is terminal in **Cat**

$\mathbf{Cat}$ has binary products

Given small categories $\mathbf{C}, \mathbf{D} \in \mathbf{Cat}$, their product $\mathbf{C} \xleftarrow{\pi_1} \mathbf{C} \times \mathbf{D} \xrightarrow{\pi_2} \mathbf{D}$ is:

$\mathbf{Cat}$ has binary products

Given small categories $\mathbf{C}, \mathbf{D} \in \mathbf{Cat}$, their product $\mathbf{C} \xleftarrow{\pi_1} \mathbf{C} \times \mathbf{D} \xrightarrow{\pi_2} \mathbf{D}$ is:

- objects of $\mathbf{C} \times \mathbf{D}$ are pairs (X, Y) where $X \in \mathbf{C}$ and $Y \in \mathbf{D}$
- morphisms $(X, Y) \to (X', Y')$ in $\mathbf{C} \times \mathbf{D}$ are pairs (f, g) where $f \in \mathbf{C}(X, X')$ and $g \in \mathbf{D}(Y, Y')$
- composition and identity morphisms are given by those of $\mathbf{C}$ (in the first component) and $\mathbf{D}$ (in the second component)
- $$\begin{cases} \pi_1\left((X, Y) \xrightarrow{(f,g)} (X', Y')\right) = X \xrightarrow{f} X' \\ \pi_2\left((X, Y) \xrightarrow{(f,g)} (X', Y')\right) = Y \xrightarrow{g} Y' \end{cases}$$

Cat not only has finite products, it is also cartesian closed.

Exponentials in **Cat** are called functor categories.

To define them we need to consider natural transformations, which are the appropriate notion of morphism between functors.

Natural transformations

Motivating example: fix a set $S \in \mathbf{Set}$ and consider the two functors $F, G : \mathbf{Set} \to \mathbf{Set}$ given by

$$F\left(X \xrightarrow{f} Y\right) = S \times X \xrightarrow{\mathtt{id}_S \times f} S \times Y$$

$$G\left(X \xrightarrow{f} Y\right) = X \times S \xrightarrow{f \times \mathtt{id}_S} Y \times S$$

Natural transformations

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For each $X \in \mathbf{Set}$ there is an isomorphism (bijection) $\theta_X : F\,X \cong G\,X$ in **Set** given by $\langle \pi_2 , \pi_1 \rangle : S \times X \to X \times S$.

These isomorphisms do not depend on the particular nature of each set X (they are "polymorphic in X"). One way to make this precise is...

...if we change from X to Y along a function $f : X \to Y$, then we get a commutative diagram in **Set**:

$$
\begin{tikzcd}
S \times X \arrow[r, "\langle \pi_2, \pi_1 \rangle"] \arrow[d, "\mathtt{id} \times f"'] & X \times S \arrow[d, "f \times \mathtt{id}"] \\
S \times Y \arrow[r, "\langle \pi_2, \pi_1 \rangle"'] & Y \times S
\end{tikzcd}
$$

The square commutes because for all $s \in S$ and $x \in X$

$$
\begin{aligned}
\langle \pi_2 , \pi_1 \rangle((\mathtt{id} \times f)(s, x)) &= \langle \pi_2 , \pi_1 \rangle(s, f\,x) \\
&= (f\,x, s) \\
&= (f \times \mathtt{id})(x, s) \\
&= (f \times \mathtt{id})(\langle \pi_2 , \pi_1 \rangle(s, x))
\end{aligned}
$$

...if we change from X to Y along a function $f : X \to Y$, then we get a commutative diagram in **Set**:

$$\begin{tikzcd} F\,X \arrow[r, "\theta_X"] \arrow[d, "F\,f"'] & G\,X \arrow[d, "G\,f"] \\ F\,Y \arrow[r, "\theta_Y"'] & G\,Y \end{tikzcd}$$

We say that the family $(\theta_X \mid X \in \textbf{Set})$ is natural in X.

Natural transformations

Definition. Given categories and functors $F, G : \mathbf{C} \to \mathbf{D}$, a natural transformation $\boxed{\theta : F \to G}$ is a family of $\mathbf{D}$-morphisms $\theta_X \in \mathbf{D}(F\,X, G\,X)$, one for each $X \in \mathbf{C}$, such that for all $\mathbf{C}$-morphisms $f : X \to Y$, the diagram

$$\begin{tikzcd} F\,X \arrow[r, "\theta_X"] \arrow[d, "F\,f"'] & G\,X \arrow[d, "G\,f"] \\ F\,Y \arrow[r, "\theta_Y"'] & G\,Y \end{tikzcd}$$

commutes in $\mathbf{D}$, that is, $\theta_Y \circ F\,f = G\,f \circ \theta_X$.

Example

Recall forgetful (U) and free (F) functors:

$$\begin{tikzcd} \mathbf{Set} \arrow[r, bend right, "F"'] & \mathbf{Mon} \arrow[l, bend right, "U"'] \end{tikzcd}$$

There is a natural transformation $\eta : \mathtt{id}_{\mathbf{Set}} \to U \circ F$, where for each $\Sigma \in \mathbf{Set}$

$$\eta_\Sigma : \Sigma \to U(F\,\Sigma) = \mathtt{List}\,\Sigma$$

$$a \in \Sigma \mapsto [a] \in \mathtt{List}\,\Sigma \quad \text{(one-element list)}$$

(Easy to see that
$$\begin{tikzcd} \Sigma \arrow[r, "\eta_\Sigma"] \arrow[d, "f"'] & U(F\,\Sigma) \arrow[d, "U(F\,f)"] \\ \Sigma' \arrow[r, "\eta_{\Sigma'}"'] & U(F\,\Sigma') \end{tikzcd}$$
commutes.)

Example

The covariant powerset functor $\mathcal{P} : \mathbf{Set} \to \mathbf{Set}$ is

$$\mathcal{P}\, X \triangleq \{S \mid S \subseteq X\}$$

$$\mathcal{P}\left(X \xrightarrow{f} Y\right) \triangleq \mathcal{P}\, X \xrightarrow{\mathcal{P}\, f} \mathcal{P}\, Y$$

$$S \mapsto \mathcal{P}\, f\, S \triangleq \{f\, x \mid x \in S\}$$

Example

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There is a natural transformation $\cup : \mathcal{P} \circ \mathcal{P} \to \mathcal{P}$ whose component at $X \in \mathbf{Set}$ sends $\mathcal{S} \in \mathcal{P}(\mathcal{P}\,X)$ to

$$\cup_X \mathcal{S} \triangleq \{x \in X \mid \exists S \in \mathcal{S},\ x \in S\} \in \mathcal{P}\,X$$

(check that $\cup_X$ is natural in X)

Non-example

The classic example of an "un-natural transformation" (the one that caused Eilenburg and MacLane to invent the concept of naturality) is the linear isomorphism between a finite dimensional real vectorspace V and its dual V^* ($=$ vectorspace of linear functions $V \to \mathbb{R}$).

Both V and V^* have the same finite dimension, so are isomorphic by choosing bases; but there is no choice of basis for each V that makes the family of isomorphisms natural in V.

For a similar, more elementary non-example, see Ex. Sh. 5, question 4.

Composing natural transformations

Given functors $F, G, H : \mathbf{C} \to \mathbf{D}$ and natural transformations $\theta : F \to G$ and $\varphi : G \to H$,

we get $\boxed{\varphi \circ \theta} : F \to H$ with

$$(\varphi \circ \theta)_X = \left(F\,X \xrightarrow{\theta_X} G\,X \xrightarrow{\varphi_X} H\,X \right)$$

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$$(\varphi \circ \theta)_X = \left(F\,X \xrightarrow{\theta_X} G\,X \xrightarrow{\varphi_X} H\,X \right)$$

Check naturality:

$$\begin{aligned}
H\,f \circ (\varphi \circ \theta)_X &\triangleq H\,f \circ \varphi_X \circ \theta_X \\
&= \varphi_Y \circ G\,f \circ \theta_X && \text{naturality of } \varphi \\
&= \varphi_Y \circ \theta_Y \circ F\,f && \text{naturality of } \theta \\
&\triangleq (\varphi \circ \theta)_Y \circ F\,f
\end{aligned}$$

Identity natural transformation

Given a functor $F : \mathbf{C} \longrightarrow \mathbf{D}$, we get a natural transformation $\boxed{\mathtt{id}_F : F \longrightarrow F}$ with

$$(\mathtt{id}_F)_X = F\,X \xrightarrow{\mathtt{id}_{F\,X}} F\,X$$

Identity natural transformation

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Check naturality:

$$F\,f \circ (\mathtt{id}_F)_X \triangleq F\,f \circ \mathtt{id}_{F\,X} = F\,f = \mathtt{id}_{F\,Y} \circ F\,f \triangleq (\mathtt{id}_F)_Y \circ F\,f$$

Functor categories

It is easy to see that composition and identities for natural transformations satisfy

$$(\psi \circ \varphi) \circ \theta = \psi \circ (\varphi \circ \theta)$$
$$\mathtt{id}_G \circ \theta = \theta \circ \mathtt{id}_F$$

so that we get a category:

Definition. Given categories $\mathbf{C}$ and $\mathbf{D}$, the functor category $\boxed{\mathbf{D}^{\mathbf{C}}}$ has

- objects are all functors $\mathbf{C} \to \mathbf{D}$
- given $F, G : \mathbf{C} \to \mathbf{D}$, morphism from F to G in $\mathbf{D}^{\mathbf{C}}$ are the natural transformations $F \to G$
- composition and identity morphisms as above

If $\mathcal{U}$ is a Grothendieck universe, then for each $X \in \mathcal{U}$ and $F \in \mathcal{U}^X$ we have that their dependent product and dependent function sets

$$\sum_{x \in X} F\,x \triangleq \{(x, y) \mid x \in X \wedge y \in F\,x\}$$
$$\prod_{x \in X} F\,x \triangleq \{f \subseteq \sum_{x \in X} F\,x \mid f \text{ is single-valued and total}\}$$

are also in $\mathcal{U}$.

Recall :

Grothendieck universes

A Grothendieck universe $\mathcal{U}$ is a set of sets satisfying

- $X \in Y \in \mathcal{U} \Rightarrow X \in \mathcal{U}$
- $X, Y \in \mathcal{U} \Rightarrow \{X, Y\} \in \mathcal{U}$
- $X \in \mathcal{U} \Rightarrow \mathcal{P}\,X \triangleq \{Y \mid Y \subseteq X\} \in \mathcal{U}$
- $X \in \mathcal{U} \wedge F \in \mathcal{U}^X \Rightarrow$
  $\{y \mid \exists x \in X,\ y \in F\,x\} \in \mathcal{U}$

  (hence also $X, Y \in \mathcal{U}\ \Rightarrow\ X \times Y \in \mathcal{U}\ \wedge\ Y^X \in \mathcal{U}$)

The above properties are satisfied by $\mathcal{U} = \emptyset$, but we will always assume

- $\mathbb{N} \in \mathcal{U}$

If $\mathcal{U}$ is a Grothendieck universe, then for each $X \in \mathcal{U}$ and $F \in \mathcal{U}^X$ we have that their dependent product and dependent function sets

$$\textstyle\sum_{x \in X} F\,x \triangleq \{(x,y) \mid x \in X \wedge y \in F\,x\}$$

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are also in $\mathcal{U}$. Hence

> If $\mathbf{C}$ and $\mathbf{D}$ are small categories, then so is $\mathbf{D}^{\mathbf{C}}$.

because

$$\mathtt{obj}(\mathbf{D}^{\mathbf{C}}) \subseteq \textstyle\sum_{F \in (\mathtt{obj}\,D)^{\mathtt{obj}\,\mathbf{C}}} \prod_{X,Y \in \mathtt{obj}\,\mathbf{C}} \mathbf{D}(F\,X, F\,Y)$$

$$\mathbf{D}^{\mathbf{C}}(F,G) \subseteq \textstyle\prod_{X \in \mathtt{obj}\,\mathbf{C}} \mathbf{D}(F\,X, G\,X)$$

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Aim to show that functor category $\mathbf{D}^{\mathbf{C}}$ is the exponential of $\mathbf{C}$ and $\mathbf{D}$ in **Cat**...

Cat is cartesian closed

Theorem. There is an application functor

$$\mathrm{app} : \mathbf{D}^{\mathbf{C}} \times \mathbf{C} \to \mathbf{D}$$

that makes $\mathbf{D}^{\mathbf{C}}$ the exponential for $\mathbf{C}$ and $\mathbf{D}$ in **Cat**.

Given $(F, X) \in \mathbf{D}^{\mathbf{C}} \times \mathbf{C}$, we define

$$\mathrm{app}(F, X) \triangleq F\,X$$

and given $(\theta, f) : (F, X) \to (G, Y)$ in $\mathbf{D}^{\mathbf{C}} \times \mathbf{C}$, we define

$$\begin{aligned} \mathrm{app}\left((F, X) \xrightarrow{(\theta, f)} (G, Y)\right) &\triangleq F\,X \xrightarrow{F\,f} F\,Y \xrightarrow{\theta_Y} G\,Y \\ &= F\,X \xrightarrow{\theta_X} G\,X \xrightarrow{G\,f} G\,Y \end{aligned}$$

Check: $\begin{cases} \mathtt{app}(\mathtt{id}_F, \mathtt{id}_X) & = \mathtt{id}_{F\,X} \\ \mathtt{app}(\varphi \circ \theta, g \circ f) & = \mathtt{app}(\varphi, g) \circ \mathtt{app}(\theta, f) \end{cases}$

$\mathbf{Cat}$ is cartesian closed

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that makes $\mathbf{D}^{\mathbf{C}}$ the exponential for $\mathbf{C}$ and $\mathbf{D}$ in $\mathbf{Cat}$.

Definition of currying: given functor $F : \mathbf{E} \times \mathbf{C} \to \mathbf{D}$, we get a functor $\mathtt{cur}\, F : \mathbf{E} \to \mathbf{D}^{\mathbf{C}}$ as follows. For each $Z \in \mathbf{E}$, $\mathtt{cur}\, F\, Z \in \mathbf{D}^{\mathbf{C}}$ is the functor

$$\mathtt{cur}\, F\, Z \left(\begin{array}{c} X \\ \downarrow f \\ X' \end{array} \right) \triangleq \begin{array}{c} F(Z, X) \\ \downarrow F(\mathtt{id}_Z, f) \\ F(Z, X') \end{array}$$

For each $g : Z \to Z'$ in $\mathbf{E}$, $\mathtt{cur}\, F\, g : \mathtt{cur}\, F\, Z \to \mathtt{cur}\, F\, Z'$ is the natural transformation whose component at each $X \in \mathbf{C}$ is

$$(\mathtt{cur}\, F\, g)_X \triangleq F(g, \mathtt{id}_X) : F(Z, X) \to F(Z', X)$$

(Check that this is natural in X; and that $\mathtt{cur}\, F$ preserves composition and identities in $\mathbf{E}$.)

$\mathbf{Cat}$ is cartesian closed

Theorem. There is an application functor

$$\mathrm{app} : \mathbf{D}^{\mathbf{C}} \times \mathbf{C} \to \mathbf{D}$$

that makes $\mathbf{D}^{\mathbf{C}}$ the exponential for $\mathbf{C}$ and $\mathbf{D}$ in $\mathbf{Cat}$.

Have to check that $\mathtt{cur}\, F$ is the unique functor $G : \mathbf{E} \to \mathbf{D}^{\mathbf{C}}$ that makes

$$\begin{tikzcd}
\mathbf{E} \times \mathbf{C} \arrow[r, "F"] \arrow[d, "G \times \mathrm{id}_{\mathbf{C}}"'] & \mathbf{D} \\
\mathbf{D}^{\mathbf{C}} \times \mathbf{C} \arrow[ur, "\mathrm{app}"'] &
\end{tikzcd}$$

commute in $\mathbf{Cat}$ (exercise).