Exercise Sheet 4
(graded, 25% of final course mark)

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BY 16:00 ON THURSDAY 15 NOVEMBER

Lecture 10

GENERAL THEORY OF NATURAL EQUIVALENCES

functor

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category

CONTENTS

Introduction. The subject matter of this paper is best explained by an example, such as that of the relation between a vector space L and its "dual"

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Functors

are the appropriate notion of morphism between categories

Given categories $\mathbf{C}$ and $\mathbf{D}$, a functor $\boxed{F : \mathbf{C} \to \mathbf{D}}$ is specified by:

- a function $\mathtt{obj}\,\mathbf{C} \to \mathtt{obj}\,\mathbf{D}$ whose value at X is written $\boxed{F\,X}$
- for all $X, Y \in \mathbf{C}$, a function $\mathbf{C}(X, Y) \to \mathbf{D}(F\,X, F\,Y)$ whose value at $f : X \to Y$ is written $\boxed{F\,f : F\,X \to F\,Y}$ and which is required to preserve composition and identity morphisms:

$$\begin{aligned} F(g \circ f) &= F\,g \circ F\,f \\ F(\mathrm{id}_X) &= \mathrm{id}_{F\,X} \end{aligned}$$

Examples of functors

"Forgetful" functors from categories of set-with-structure back to **Set**.

E.g. $\boxed{U : \mathbf{Mon} \to \mathbf{Set}}$

$$\begin{cases} U(M, \cdot, e) & = M \\ U((M_1, \cdot_1, e_1) \xrightarrow{f} (M_2, \cdot_2, e_2)) & = M_1 \xrightarrow{f} M_2 \end{cases}$$

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Similarly $U : \mathbf{Preord} \to \mathbf{Set}$.

Examples of functors

Free monoid functor $\boxed{F : \mathbf{Set} \longrightarrow \mathbf{Mon}}$

Given $\mathbf{\Sigma} \in \mathbf{Set}$,

$$F\,\mathbf{\Sigma} = (\mathtt{List}\,\mathbf{\Sigma}, @, \mathtt{nil}), \text{ the free monoid on } \mathbf{\Sigma}$$

Examples of functors

Free monoid functor $\boxed{F : \mathbf{Set} \to \mathbf{Mon}}$

Given $\Sigma \in \mathbf{Set}$,

$$F\,\Sigma = (\mathtt{List}\,\Sigma, @, \mathtt{nil}), \text{ the free monoid on } \Sigma$$

Given a function $f : \Sigma_1 \to \Sigma_2$, we get a function $F\,f : \mathtt{List}\,\Sigma_1 \to \mathtt{List}\,\Sigma_2$ by mapping f over finite lists:

$$F\,f\,[a_1, \ldots, a_n] = [f\,a_1, \ldots, f\,a_n]$$

This gives a monoid morphism $F\,\Sigma_1 \to F\,\Sigma_2$; and mapping over lists preserves composition ($F(g \circ f) = F\,g \circ F\,f$) and identities ($F\,\mathrm{id}_\Sigma = \mathrm{id}_{F\,\Sigma}$). So we do get a functor from **Set** to **Mon**.

Examples of functors

If $\mathbf{C}$ is a category with binary products and $X \in \mathbf{C}$, then the function $(\_) \times X : \mathtt{obj}\,\mathbf{C} \to \mathtt{obj}\,\mathbf{C}$ extends to a functor $\boxed{(\_) \times X : \mathbf{C} \to \mathbf{C}}$ mapping morphisms $f : Y \to Y'$ to

$$f \times \mathtt{id}_X : Y \times X \to Y' \times X$$

$$\left(\text{recall that } f \times g \text{ is the unique morphism with } \begin{cases} \mathtt{fst} \circ (f \times g) & = f \circ \mathtt{fst} \\ \mathtt{snd} \circ (f \times g) & = g \circ \mathtt{snd} \end{cases}\right)$$

since it is the case that

$$\begin{cases} \mathtt{id}_X \times \mathtt{id}_Y & = \mathtt{id}_{X \times Y} \\ (f' \circ f) \times \mathtt{id}_X & = (f' \times \mathtt{id}_X) \circ (f \times \mathtt{id}_X) \end{cases}$$

(see Exercise Sheet 2, question 1c).

Examples of functors

If $\mathbf{C}$ is a cartesian closed category and $X \in \mathbf{C}$, then the function $(\_)^X : \mathtt{obj}\,\mathbf{C} \to \mathtt{obj}\,\mathbf{C}$ extends to a functor $\boxed{(\_)^X : \mathbf{C} \to \mathbf{C}}$ mapping morphisms $f : Y \to Y'$ to

$$f^X \triangleq \mathtt{cur}(f \circ \mathtt{app}) : Y^X \to {Y'}^X$$

since it is the case that $\begin{cases} (\mathtt{id}_Y)^X &= \mathtt{id}_{Y^X} \\ (g \circ f)^X &= g^X \circ f^X \end{cases}$

(see Exercise Sheet 3, question 4).

Contravariance

Given categories $\mathbf{C}$ and $\mathbf{D}$, a functor $F : \mathbf{C}^{\mathrm{op}} \longrightarrow \mathbf{D}$ is called a contravariant functor from $\mathbf{C}$ to $\mathbf{D}$.

Note that if $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathbf{C}$, then $X \xleftarrow{f} Y \xleftarrow{g} Z$ in $\mathbf{C}^{\mathrm{op}}$

so $F\,X \xleftarrow{F\,f} F\,Y \xleftarrow{F\,g} F\,Z$ in $\mathbf{D}$ and hence

$$\boxed{F(g \circ_{\mathbf{C}} f) = F\,f \circ_{\mathbf{D}} F\,g}$$

(contravariant functors reverse the order of composition)

A functor $\mathbf{C} \rightarrow \mathbf{D}$ is sometimes called a covariant functor from $\mathbf{C}$ to $\mathbf{D}$.

Example of a contravariant functor

If $\mathbf{C}$ is a cartesian closed category and $X \in \mathbf{C}$, then the function $X^{(-)} : \mathtt{obj}\,\mathbf{C} \to \mathtt{obj}\,\mathbf{C}$ extends to a functor $\boxed{X^{(-)} : \mathbf{C}^{\mathrm{op}} \to \mathbf{C}}$ mapping morphisms $f : Y \to Y'$ to

$$X^f \triangleq \mathtt{cur}(\mathtt{app} \circ (\mathtt{id}_{X^{Y'}} \times f)) : X^{Y'} \to X^Y$$

since it is the case that $\begin{cases} X^{\mathtt{id}_Y} & = \mathtt{id}_{X^Y} \\ X^{g \circ f} & = X^f \circ X^g \end{cases}$

(see Exercise Sheet 3, question 5).

Note that since a functor $F : \mathbf{C} \longrightarrow \mathbf{D}$ preserves domains, codomains, composition and identity morphisms

it sends commutative diagrams in $\mathbf{C}$ to commutative diagrams in $\mathbf{D}$

E.g.

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[dd, "h"] \\ Y \arrow[dr, "g"'] & \\ & Z \end{tikzcd} \quad \overset{F}{\longmapsto} \quad \begin{tikzcd} & F\,X \arrow[dl, "F\,f"'] \arrow[dd, "F\,h = F(g \circ f) = F\,g \circ F\,f"] \\ F\,Y \arrow[dr, "F\,g"'] & \\ & F\,Z \end{tikzcd}$$

Note that since a functor $F : \mathbf{C} \longrightarrow \mathbf{D}$ preserves domains, codomains, composition and identity morphisms

it sends isomorphisms in $\mathbf{C}$ to isomorphisms in $\mathbf{D}$, because

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[dr, "\mathrm{id}_X"'] & Y \arrow[d, "g"] \arrow[dr, "\mathrm{id}_Y"] & \\
& X \arrow[r, "f"'] & Y
\end{tikzcd}
\quad \overset{F}{\longmapsto} \quad
\begin{tikzcd}
F\,X \arrow[r, "F\,f"] \arrow[dr, "\mathrm{id}_{F\,X}"'] & F\,Y \arrow[d, "F\,g"] \arrow[dr, "\mathrm{id}_{F\,Y}"] & \\
& F\,X \arrow[r, "F\,f"'] & F\,Y
\end{tikzcd}
$$

so $\boxed{F(f^{-1}) = (F\,f)^{-1}}$

Composing functors

Given functors $F : \mathbf{C} \longrightarrow \mathbf{D}$ and $G : \mathbf{D} \longrightarrow \mathbf{E}$, we get a functor $\boxed{G \circ F : \mathbf{C} \longrightarrow \mathbf{E}}$ with

$$G \circ F \left(\begin{array}{c} X \\ \downarrow f \\ Y \end{array} \right) = \begin{array}{c} G(F\,X) \\ \downarrow G(F\,f) \\ G(F\,Y) \end{array}$$

(this preserves composition and identity morphisms, because F and G do)

Identity functor

on a category $\mathbf{C}$ is $\boxed{\mathtt{id}_{\mathbf{C}} : \mathbf{C} \to \mathbf{C}}$ where

$$\mathtt{id}_{\mathbf{C}} \left(\begin{array}{c} X \\ \downarrow f \\ Y \end{array} \right) = \begin{array}{c} X \\ \downarrow f \\ Y \end{array}$$

Functor composition and identity functors satisfy

associativity	$H \circ (G \circ F) = (H \circ G) \circ F$
unity	$\mathtt{id}_{\mathbf{D}} \circ F = F = F \circ \mathtt{id}_{\mathbf{C}}$

So we can get categories whose objects are categories and whose morphisms are functors

but we have to be a bit careful about size. . .

Size

One of the axioms of set theory is

set membership is a well-founded relation, that is, there is no infinite sequence of sets $X_0, X_1, X_2, \ldots$ with

$$\cdots \in X_{n+1} \in X_n \in \cdots \in X_2 \in X_1 \in X_0$$

So in particular there is no set X with $X \in X$.

So we cannot form the "set of all sets" or the "category of all categories".

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So in particular there is no set X with $X \in X$.

So we cannot form the "set of all sets" or the "category of all categories".

But we do assume there are (lots of) big sets

$$\mathcal{U}_0 \in \mathcal{U}_1 \in \mathcal{U}_2 \in \cdots$$

where "big" means each $\mathcal{U}_n$ is a Grothendieck universe...

Grothendieck universes

A Grothendieck universe $\mathcal{U}$ is a set of sets satisfying

- $X \in Y \in \mathcal{U} \Rightarrow X \in \mathcal{U}$
- $X, Y \in \mathcal{U} \Rightarrow \{X, Y\} \in \mathcal{U}$
- $X \in \mathcal{U} \Rightarrow \mathcal{P}\,X \triangleq \{Y \mid Y \subseteq X\} \in \mathcal{U}$
- $X \in \mathcal{U} \wedge F \in \mathcal{U}^X \Rightarrow$
  $\{y \mid \exists x \in X,\; y \in F\,x\} \in \mathcal{U}$

  (hence also $X, Y \in \mathcal{U} \;\Rightarrow\; X \times Y \in \mathcal{U} \;\wedge\; Y^X \in \mathcal{U}$)

The above properties are satisfied by $\mathcal{U} = \emptyset$, but we will always assume

- $\mathbb{N} \in \mathcal{U}$

Size

We assume

there is an infinite sequence $\mathcal{U}_0 \in \mathcal{U}_1 \in \mathcal{U}_2 \in \cdots$ of bigger and bigger Grothendieck universes

and revise the previous definition of "the" category of sets and functions:

$\mathbf{Set}_n$ = category whose objects are all the sets in $\mathcal{U}_n$ and with $\mathbf{Set}_n(X, Y) = Y^X =$ all functions from X to Y.

Notation: $\boxed{\mathbf{Set} \triangleq \mathbf{Set}_0}$ — its objects are called small sets (and other sets we call large).

Size

Set is the category of small sets.

Definition. A category **C** is locally small if for all $X, Y \in \mathbf{C}$, the set of **C**-morphisms $X \to Y$ is small, that is, $\mathbf{C}(X, Y) \in \mathbf{Set}$.

C is a small category if it is both locally small and $\mathtt{obj}\,\mathbf{C} \in \mathbf{Set}$.

E.g. **Set**, **Preord** and **Mon** are all locally small (but not small).

Given $P \in \mathbf{Preord}$, the cateogry $\mathbf{C}_P$ it determines is small; similarly, the category $\mathbf{C}_M$ determined by $M \in \mathbf{Mon}$ is small.

The category of small categories, $\mathbf{Cat}$

- objects are all small categories
- morphisms in $\mathbf{Cat}(\mathbf{C},\mathbf{D})$ are all functors $\mathbf{C} \to \mathbf{D}$
- composition and identity morphisms as for functors

$\mathbf{Cat}$ is a locally small category