Exercise Sheet 4 (graded, 25% of final course mark)

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Lecture 10

GENERAL THEORY OF NATURAL EQUIVALENCES

BY

SAMUEL EILENBERG AND SAUNDERS MACLANE

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Introduction. The subject matter of this paper is best explained by an example, such as that of the relation between a vector space L and its "dual".

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functor Lakgory

Functors

are the appropriate notion of morphism between categories

Given categories C and D, a functor $F: C \rightarrow D$ is specified by:

- ▶ a function obj $C \rightarrow obj D$ whose value at X is written FX
- For all X, Y ∈ C, a function
 C(X,Y) → D(FX, FY) whose value at
 f: X → Y is written Ff: FX → FY
 and which is required to preserve composition and
 identity morphisms:

 $\begin{array}{rcl}F(g \circ f) &=& F \, g \circ F \, f \\F(\mathrm{id}_X) &=& \mathrm{id}_{F \, X}\end{array}$

"Forgetful" functors from categories of set-with-structure back to **Set**.

E.g. $U: Mon \rightarrow Set$

$$\begin{cases} U(M, \cdot, e) &= M \\ U((M_1, \cdot_1, e_1) \xrightarrow{f} (M_2, \cdot_2, e_2)) &= M_1 \xrightarrow{f} M_2 \end{cases}$$

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Similarly $U : \mathsf{Preord} \to \mathsf{Set}$.

Free monoid functor $F: Set \rightarrow Mon$ Given $\Sigma \in Set$,

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Free monoid functor $F : Set \rightarrow Mon$ Given $\Sigma \in Set$,

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Given a function $f: \Sigma_1 \to \Sigma_2$, we get a function $F f: \text{List} \Sigma_1 \to \text{List} \Sigma_2$ by mapping f over finite lists:

$$F f [a_1,\ldots,a_n] = [f a_1,\ldots,f a_n]$$

This gives a monoid morphism $F \Sigma_1 \to F \Sigma_2$; and mapping over lists preserves composition $(F(g \circ f) = Fg \circ Ff)$ and identities $(F \operatorname{id}_{\Sigma} = \operatorname{id}_{F\Sigma})$. So we do get a functor from **Set** to **Mon**.

If **C** is a category with binary products and $X \in C$, then the function (_) × X : obj **C** \rightarrow obj **C** extends to a functor (_) × X : **C** \rightarrow **C** mapping morphisms $f: Y \rightarrow Y'$ to

 $f \times \operatorname{id}_X : Y \times X \to Y' \times X$

 $\begin{pmatrix} \text{recall that } f \times g \text{ is the unique morphism with } \begin{cases} \texttt{fst} \circ (f \times g) &= f \circ \texttt{fst} \\ \texttt{snd} \circ (f \times g) &= g \circ \texttt{snd} \end{pmatrix}$

since it is the case that $\begin{cases} \operatorname{id}_X \times \operatorname{id}_Y &= \operatorname{id}_{X \times Y} \\ (f' \circ f) \times \operatorname{id}_X &= (f' \times \operatorname{id}_X) \circ (f \times \operatorname{id}_X) \end{cases}$

(see Exercise Sheet 2, question 1c).

If **C** is a cartesian closed category and $X \in \mathbf{C}$, then the function $(_)^X : \operatorname{obj} \mathbf{C} \to \operatorname{obj} \mathbf{C}$ extends to a functor $(_)^X : \mathbf{C} \to \mathbf{C}$ mapping morphisms $f : Y \to Y'$ to

$$f^X \triangleq \operatorname{cur}(f \circ \operatorname{app}) : Y^X \to {Y'}^X$$

since it is the case that $\begin{cases} (\operatorname{id}_Y)^X &= \operatorname{id}_{Y^X} \\ (g \circ f)^X &= g^X \circ f^X \end{cases}$

(see Exercise Sheet 3, question 4).

Contravariance

Given categories C and D, a functor $F : C^{op} \rightarrow D$ is called a contravariant functor from C to D.

Note that if $X \xrightarrow{f} Y \xrightarrow{g} Z$ in **C**, then $X \xleftarrow{f} Y \xleftarrow{g} Z$ in **C**^{op} so $FX \xleftarrow{Ff} FY \xleftarrow{Fg} FZ$ in **D** and hence

$$F(g \circ_{\mathsf{C}} f) = F f \circ_{\mathsf{D}} F g$$

(contravariant functors reverse the order of composition)

A functor $\mathbf{C} \rightarrow \mathbf{D}$ is sometimes called a covariant functor from \mathbf{C} to \mathbf{D} .

Example of a contravariant functor

If **C** is a cartesian closed category and $X \in C$, then the function $X^{(-)} : \operatorname{obj} C \to \operatorname{obj} C$ extends to a functor $X^{(-)} : C^{\operatorname{op}} \to C$ mapping morphisms $f : Y \to Y'$ to

$$X^f riangleq ext{cur}(ext{app} \circ (ext{id}_{X^{Y'}} imes f)): X^{Y'} o X^Y$$

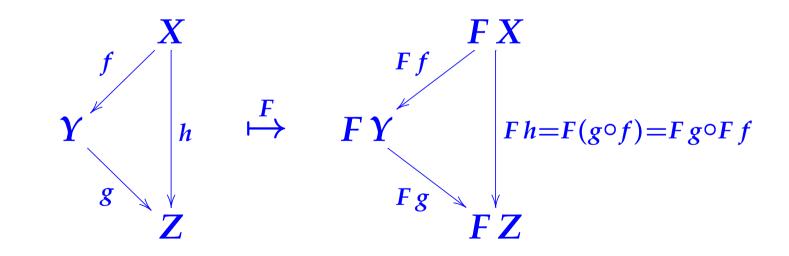
since it is the case that
$$\begin{cases} X^{\operatorname{id}_Y} &= \operatorname{id}_{X^Y} \\ X^{g \circ f} &= X^f \circ X^g \end{cases}$$

(see Exercise Sheet 3, question 5).

Note that since a functor $F : C \rightarrow D$ preserves domains, codomains, composition and identity morphisms

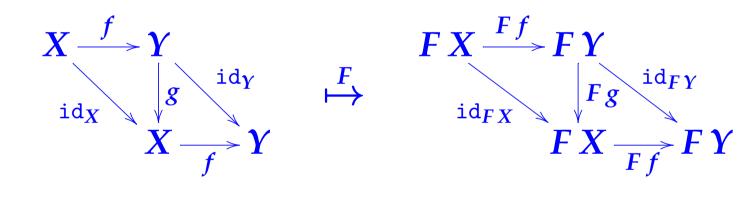
it sends commutative diagrams in ${\bf C}$ to commutative diagrams in ${\bf D}$

E.g.



Note that since a functor $F : C \rightarrow D$ preserves domains, codomains, composition and identity morphisms

it sends isomorphisms in **C** to isomorphisms in **D**, because



so
$$F(f^{-1}) = (Ff)^{-1}$$

Composing functors

Given functors $F : \mathbb{C} \to \mathbb{D}$ and $G : \mathbb{D} \to \mathbb{E}$, we get a functor $G \circ F : \mathbb{C} \to \mathbb{E}$ with

$$G \circ F \begin{pmatrix} X \\ \downarrow f \\ Y \end{pmatrix} = \begin{array}{c} G(F X) \\ \downarrow G(F f) \\ G(F Y) \end{array}$$

(this preserves composition and identity morphisms, because F and G do)

on a category **C** is $\operatorname{id}_{\mathbf{C}} : \mathbf{C} \to \mathbf{C}$ where $\operatorname{id}_{\mathbf{C}} \begin{pmatrix} X \\ \downarrow f \end{pmatrix} = \bigvee_{\mathbf{Y}}^{X}$

Functor composition and identity functors satisfy

associativity $H \circ (G \circ F) = (H \circ G) \circ F$ unity $\mathrm{id}_{\mathsf{D}} \circ F = F = F \circ \mathrm{id}_{\mathsf{C}}$

So we can get categories whose objects are categories and whose morphisms are functors

but we have to be a bit careful about size...

Size

One of the axioms of set theory is

set membership is a well-founded relation, that is, there is no infinite sequence of sets X_0, X_1, X_2, \ldots with

 $\cdots \in X_{n+1} \in X_n \in \cdots \in X_2 \in X_1 \in X_0$

So in particular there is no set X with $X \in X$.

So we cannot form the "set of all sets" or the "category of all categories".

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So in particular there is no set X with $X \in X$.

So we cannot form the "set of all sets" or the "category of all categories".

But we do assume there are (lots of) big sets

 $\mathfrak{U}_0 \in \mathfrak{U}_1 \in \mathfrak{U}_2 \in \cdots$

where "big" means each \mathcal{U}_n is a Grothendieck universe...

Grothendieck universes

A Grothendieck universe \mathcal{U} is a set of sets satisfying

The above properties are satisfied by $\mathcal{U} = \emptyset$, but we will always assume

\blacktriangleright $\mathbb{N} \in \mathfrak{U}$

Size

We assume

there is an infinite sequence $\mathcal{U}_0 \in \mathcal{U}_1 \in \mathcal{U}_2 \in \cdots$ of bigger and bigger Grothendieck universes

and revise the previous definition of "the" category of sets and functions:

Set_n = category whose objects are all the sets in \mathcal{U}_n and with Set_n(X, Y) = Y^X = all functions from X to Y.

Notation: $Set \triangleq Set_0$ — its objects are called small sets (and other sets we call large).

Size

Set is the category of small sets.

Definition. A category **C** is locally small if for all $X, Y \in C$, the set of **C**-morphisms $X \to Y$ is small, that is, $C(X, Y) \in Set$.

C is a small category if it is both locally small and obj $C \in Set$.

E.g. Set, Preord and Mon are all locally small (but not small).

Given $P \in \mathbf{Preord}$, the cateogry C_P it determines is small; similarly, the category C_M determined by $M \in \mathbf{Mon}$ is small.

The category of small categories, Cat

objects are all small categories

- ▶ morphisms in Cat(C, D) are all functors $C \rightarrow D$
- composition and identity morphisms as for functors

Cat is a locally small category