Lecture 9
STLC equations

take the form $\Gamma \vdash s = t : A$ where $\Gamma \vdash s : A$ and $\Gamma \vdash t : A$ are provable.

Such an equation is satisfied by the semantics in a ccc if $M[\Gamma \vdash s : A]$ and $M[\Gamma \vdash t : A]$ are equal $\mathbf{C}$-morphisms $M[\Gamma] \rightarrow M[A]$.

Qu: which equations are always satisfied in any ccc?
Ans: $\beta\eta$-equivalence...
STLC $\beta\eta$-Equality

The relation $\Gamma \vdash s =_{\beta\eta} t : A$ (where $\Gamma$ ranges over typing environments, $s$ and $t$ over terms and $A$ over types) is inductively defined by the following rules:
STLC $\beta\eta$-Equality

The relation $\Gamma \vdash s =_{\beta\eta} t : A$ (where $\Gamma$ ranges over typing environments, $s$ and $t$ over terms and $A$ over types) is inductively defined by the following rules:

**$\beta$-conversions**

\[
\begin{align*}
\Gamma, x : A &\vdash t : B & \Gamma &\vdash s : A \\
\Gamma \vdash (\lambda x : A. t)s & =_{\beta\eta} t[s/x] : B \\
\Gamma \vdash s : A &\quad \Gamma \vdash t : B \\
\Gamma &\vdash \text{fst}(s, t) =_{\beta\eta} s : A \\
\Gamma \vdash \text{snd}(s, t) & =_{\beta\eta} t : B
\end{align*}
\]
STLC $\beta\eta$-Equality

The relation $\Gamma \vdash s =_{\beta\eta} t : A$ (where $\Gamma$ ranges over typing environments, $s$ and $t$ over terms and $A$ over types) is inductively defined by the following rules:

- $\beta$-conversions

- $\eta$-conversions

\[
\begin{align*}
\Gamma \vdash t : A \to B & \quad x \text{ does not occur in } t \\
\hline
\Gamma \vdash t =_{\beta\eta} (\lambda x : A. t x) : A \to B
\end{align*}
\]

\[
\begin{align*}
\Gamma \vdash t : A \times B \\
\hline
\Gamma \vdash t =_{\beta\eta} (\text{fst } t, \text{snd } t) : A \times B
\end{align*}
\]

\[
\begin{align*}
\Gamma \vdash t : \text{unit} \\
\hline
\Gamma \vdash t =_{\beta\eta} () : \text{unit}
\end{align*}
\]
The relation $\Gamma \vdash s =_{\beta\eta} t : A$ (where $\Gamma$ ranges over typing environments, $s$ and $t$ over terms and $A$ over types) is inductively defined by the following rules:

- **$\beta$-conversions**
  
- **$\eta$-conversions**

- **congruence rules**

\[
\begin{align*}
\Gamma, x : A & \vdash t =_{\beta\eta} t' : B \\
\Gamma & \vdash \lambda x : A. t =_{\beta\eta} \lambda x : A. t' : A \to B \\
\Gamma & \vdash s =_{\beta\eta} s' : A \to B \quad \Gamma & \vdash t =_{\beta\eta} t' : A \\
\Gamma & \vdash s \, t =_{\beta\eta} s' \, t' : B
\end{align*}
\]
STLC $\beta\eta$-Equality

The relation $\Gamma \vdash s =_{\beta\eta} t : A$ (where $\Gamma$ ranges over typing environments, $s$ and $t$ over terms and $A$ over types) is inductively defined by the following rules:

- $\beta$-conversions
- $\eta$-conversions
- congruence rules
- $=_{\beta\eta}$ is reflexive, symmetric and transitive

\[
\begin{array}{c}
\Gamma \vdash t : A \\
\hline
\Gamma \vdash t =_{\beta\eta} t : A
\end{array}
\quad
\begin{array}{c}
\Gamma \vdash s =_{\beta\eta} t : A \\
\hline
\Gamma \vdash t =_{\beta\eta} s : A
\end{array}
\quad
\begin{array}{c}
\Gamma \vdash r =_{\beta\eta} s : A \\
\hline
\Gamma \vdash r =_{\beta\eta} t : A
\end{array}
\quad
\begin{array}{c}
\Gamma \vdash s =_{\beta\eta} t : A \\
\hline
\Gamma \vdash r =_{\beta\eta} t : A
\end{array}
\]
Soundness Theorem for semantics of STLC in a ccc. If $\Gamma \vdash s \approx_{\beta\eta} t : A$ is provable, then in any ccc

$$M[\Gamma \vdash s : A] = M[\Gamma \vdash t : A]$$

are equal $\mathsf{C}$-morphisms $M[\Gamma] \to M[A]$.

Proof is by induction on the structure of the proof of $\Gamma \vdash s \approx_{\beta\eta} t : A$.

Here we just check the case of $\beta$-conversion for functions.

So suppose we have $\Gamma, x : A \vdash t : B$ and $\Gamma \vdash s : A$. We have to see that

$$M[\Gamma \vdash (\lambda x : A. t) s : B] = M[\Gamma \vdash t[s/x] : B]$$
Suppose

\[ M[\Gamma] = X \]
\[ M[A] = Y \]
\[ M[B] = Z \]
\[ M[\Gamma, x : A \vdash t : B] = f : X \times Y \to Z \]
\[ M[\Gamma \vdash s : A] = g : X \to Z \]

Then

\[ M[\Gamma \vdash \lambda x : A. t : A \to B] = \text{cur } f : X \to Z^Y \]

and hence

\[ M[\Gamma \vdash (\lambda x : A. t)s : B] \]
\[ = \text{app } \circ \langle \text{cur } f, g \rangle \]
\[ = \text{app } \circ (\text{cur } f \times \text{id}_Y) \circ \langle \text{id}_X, g \rangle \quad \text{since } (a \times b) \circ \langle c, d \rangle = \langle a \circ c, b \circ d \rangle \]
\[ = f \circ \langle \text{id}_X, g \rangle \quad \text{by definition of } \text{cur } f \]
\[ = M[\Gamma \vdash t[s/x] : B] \quad \text{by the Substitution Theorem} \]

as required.
The internal language of a ccc, \( \mathcal{C} \)

- one ground type for each \( \mathcal{C} \)-object \( X \)
- for each \( X \in \mathcal{C} \), one constant \( f^X \) for each \( \mathcal{C} \)-morphism \( f : 1 \to X \) (“global element” of the object \( X \))

The types and terms of STLC over this language usefully describe constructions on the objects and morphisms of \( \mathcal{C} \) using its cartesian closed structure, but in an “element-theoretic” way.

For example...
Example

In any ccc $C$, for any $X, Y, Z \in C$ there is an isomorphism

$$Z^{(X \times Y)} \cong (Z^Y)^X$$
Example

In any ccc $\mathbf{C}$, for any $X, Y, Z \in \mathbf{C}$ there is an isomorphism

$$Z^{(X \times Y)} \cong (Z^Y)^X$$

which in the internal language of $\mathbf{C}$ is described by the terms

$$\diamond \vdash s : ((X \times Y) \to Z) \to (X \to (Y \to Z))$$
$$\diamond \vdash t : (X \to (Y \to Z)) \to ((X \times Y) \to Z)$$

where

$$\begin{cases} s \triangleq \lambda f : (X \times Y) \to Z. \lambda x : X. \lambda y : Y. f(x, y) \\ t \triangleq \lambda g : X \to (Y \to Z). \lambda z : X \times Y. g(fst\ z)\ (snd\ z) \end{cases}$$

and which satisfy

$$\begin{cases} \diamond, f : (X \times Y) \to Z \vdash t(sf) =_{\beta\eta} f \\ \diamond, g : X \to (Y \to Z) \vdash s(tg) =_{\beta\eta} g \end{cases}$$
Free cartesian closed categories

The **Soundness Theorem** has a converse—completeness.

In fact for a given set of ground types and typed constants there is a single ccc $\mathbf{F}$ (the **free ccc** for that language) with an interpretation function $M$ so that $\Gamma \vdash s =_{\beta\eta} t : A$ is provable iff $M[\Gamma \vdash s : A] = M[\Gamma \vdash t : A]$ in $\mathbf{F}$.
Free cartesian closed categories

The Soundness Theorem has a converse—completeness.

In fact for a given set of ground types and typed constants there is a single ccc \( \mathbf{F} \) (the free ccc for that language) with an interpretation function \( M \) so that \( \Gamma \vdash s = \beta\eta t : A \) is provable iff \( M[\Gamma \vdash s : A] = M[\Gamma \vdash t : A] \) in \( \mathbf{F} \).

- **\( \mathbf{F} \)-objects** are the STLC types over the given set of ground types.

- **\( \mathbf{F} \)-morphisms** \( A \to B \) are equivalence classes of STLC terms \( t \) satisfying \( \diamond \vdash t : A \to B \) (so \( t \) is a closed term—it has no free variables) with respect to the equivalence relation equating \( s \) and \( t \) if \( \diamond \vdash s = \beta\eta t : A \to B \) is provable.

- identity morphism on \( A \) is the equivalence class of \( \diamond \vdash \lambda x : A. x : A \to A \).

- composition of a morphism \( A \to B \) represented by \( \diamond \vdash s : A \to B \) and a morphism \( B \to C \) represented by \( \diamond \vdash t : B \to C \) is represented by \( \diamond \vdash \lambda x : A. t(s x) : A \to C \).
## Curry-Howard correspondence

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E.g. IPL *versus* STLC.
Curry-Howard for IPL vs STLC

Proof of $\Diamond, \varphi \Rightarrow \psi, \psi \Rightarrow \theta \vdash \varphi \Rightarrow \theta$ in IPL

where $\Phi = \Diamond, \varphi \Rightarrow \psi, \psi \Rightarrow \theta, \varphi$
Curry-Howard for IPL vs STLC

and a corresponding STLC term

\[ \begin{array}{c}
\vdash \Phi \vdash z : \psi \Rightarrow \theta \\
\vdash \Phi \vdash y : \phi \Rightarrow \psi \\
\vdash \Phi \vdash x : \phi (\Rightarrow E) \\
\vdash \Phi \vdash \lambda x : \phi. z(y x) : \varphi \Rightarrow \theta (\Rightarrow I) \\
\vdash \Diamond, y : \phi \Rightarrow \psi, z : \psi \Rightarrow \theta, x : \phi
\end{array} \]

where \( \Phi = \Diamond, y : \phi \Rightarrow \psi, z : \psi \Rightarrow \theta, x : \phi \)
Curry-Howard-Lawvere/Lambek correspondence

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E.g. IPL *versus* STLC *versus* CCCs
Curry-Howard-Lawvere/Lambek correspondence

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E.g. IPL *versus* STLC *versus* CCCs

These correspondences can be made into category-theoretic equivalences—we first need to define the notions of functor and natural transformation in order to define the notion of equivalence of categories.