

Lecture 7

Solutions to Exercise Sheet 2 - handout in this class

Exercise Sheet 3 now on-line + handout in this class

Need help? - office hours: Fridays 09:00-10:00 in FC08

IPL entailment $\Phi \vdash \varphi$

Recall the rules:

$\frac{}{\Phi, \varphi \vdash \varphi} \text{ (AX)}$	$\frac{\Phi \vdash \varphi}{\Phi, \psi \vdash \varphi} \text{ (WK)}$	$\frac{\Phi \vdash \varphi \quad \Phi, \varphi \vdash \psi}{\Phi \vdash \psi} \text{ (CUT)}$
$\frac{}{\Phi \vdash \text{true}} \text{ (TRUE)}$	$\frac{\Phi \vdash \varphi \quad \Phi \vdash \psi}{\Phi \vdash \varphi \& \psi} \text{ (&I)}$	$\frac{\Phi, \varphi \vdash \psi}{\Phi \vdash \varphi \Rightarrow \psi} \text{ (}\Rightarrow\text{I)}$
$\frac{\Phi \vdash \varphi \& \psi}{\Phi \vdash \varphi} \text{ (&E}_1\text{)}$	$\frac{\Phi \vdash \varphi \& \psi}{\Phi \vdash \psi} \text{ (&E}_2\text{)}$	$\frac{\Phi \vdash \varphi \Rightarrow \psi \quad \Phi \vdash \varphi}{\Phi \vdash \psi} \text{ (}\Rightarrow\text{E)}$

Proof theory

Two IPL proofs of $\diamond, \varphi \Rightarrow \psi, \psi \Rightarrow \theta \vdash \varphi \Rightarrow \theta$

$$\frac{\frac{\frac{\dots}{\Phi, \varphi \vdash \psi \Rightarrow \theta} \text{(AX)}}{\dots} \text{(WK)} \quad \frac{\frac{\frac{\dots}{\Phi, \varphi \vdash \varphi \Rightarrow \psi} \text{(AX)}}{\dots} \text{(WK)} \quad \frac{\dots}{\Phi, \varphi \vdash \varphi} \text{(AX)}}{\Phi, \varphi \vdash \psi} \text{(}\Rightarrow\text{E)}}{\Phi, \varphi \vdash \psi} \text{(}\Rightarrow\text{E)}} \quad \frac{\Phi, \varphi \vdash \theta}{\Phi \vdash \varphi \Rightarrow \theta} \text{(}\Rightarrow\text{I)}$$

where $\Phi \triangleq \diamond, \varphi \Rightarrow \psi, \psi \Rightarrow \theta$

$$\frac{\frac{\frac{\dots}{\Psi \vdash \varphi \Rightarrow \psi} \text{(AX)}}{\dots} \text{(WK)} \quad \frac{\dots}{\Psi \vdash \varphi} \text{(AX)}}{\Psi \vdash \psi} \text{(}\Rightarrow\text{E)}} \quad \frac{\frac{\frac{\dots}{\Psi, \psi \vdash \psi \Rightarrow \theta} \text{(AX)}}{\dots} \text{(WK)} \quad \frac{\dots}{\Psi, \psi \vdash \psi} \text{(AX)}}{\Psi, \psi \vdash \theta} \text{(}\Rightarrow\text{E)}}{\Psi \vdash \theta} \text{(CUT)} \quad \frac{\Psi \vdash \theta}{\diamond, \varphi \Rightarrow \psi, \psi \Rightarrow \theta \vdash \varphi \Rightarrow \theta} \text{(}\Rightarrow\text{I)}$$

where $\Psi \triangleq \diamond, \varphi \Rightarrow \psi, \psi \Rightarrow \theta, \varphi$

Proof theory

Two IPL proofs of $\diamond, \varphi \Rightarrow \psi, \psi \Rightarrow \theta \vdash \varphi \Rightarrow \theta$

$$\frac{\frac{\frac{\dots}{\Phi, \varphi \vdash \psi \Rightarrow \theta} \text{(AX)}}{\dots} \text{(WK)} \quad \frac{\frac{\frac{\dots}{\Phi, \varphi \vdash \varphi \Rightarrow \psi} \text{(AX)}}{\dots} \text{(WK)} \quad \frac{\frac{\dots}{\Phi, \varphi \vdash \varphi} \text{(AX)}}{\Phi, \varphi \vdash \psi} \text{(}\Rightarrow\text{E)}}{\Phi, \varphi \vdash \theta} \text{(}\Rightarrow\text{E)}}{\Phi \vdash \varphi \Rightarrow \theta} \text{(}\Rightarrow\text{I)}$$

where $\Phi \triangleq \diamond, \varphi \Rightarrow \psi, \psi \Rightarrow \theta$

$$\frac{\frac{\frac{\dots}{\Psi \vdash \varphi \Rightarrow \psi} \text{(AX)}}{\dots} \text{(WK)} \quad \frac{\frac{\dots}{\Psi \vdash \varphi} \text{(AX)}}{\Psi \vdash \psi} \text{(}\Rightarrow\text{E)} \quad \frac{\frac{\frac{\dots}{\Psi, \psi \vdash \psi \Rightarrow \theta} \text{(AX)}}{\dots} \text{(WK)} \quad \frac{\frac{\dots}{\Psi, \psi \vdash \psi} \text{(AX)}}{\Psi, \psi \vdash \theta} \text{(}\Rightarrow\text{E)}}{\Psi \vdash \theta} \text{(CUT)}}{\diamond, \varphi \Rightarrow \psi, \psi \Rightarrow \theta \vdash \varphi \Rightarrow \theta} \text{(}\Rightarrow\text{I)}$$

where $\Psi \triangleq \diamond, \varphi \Rightarrow \psi, \psi \Rightarrow \theta, \varphi$

Why is the first proof simpler than the second one?

Proof theory

$\frac{}{\Phi, \varphi \vdash \varphi} \text{ (AX)}$	$\frac{\Phi \vdash \varphi}{\Phi, \psi \vdash \varphi} \text{ (WK)}$	$\frac{\Phi \vdash \varphi \quad \Phi, \varphi \vdash \psi}{\Phi \vdash \psi} \text{ (CUT)}$
$\frac{}{\Phi \vdash \text{true}} \text{ (TRUE)}$	$\frac{\Phi \vdash \varphi \quad \Phi \vdash \psi}{\Phi \vdash \varphi \& \psi} \text{ (&I)}$	$\frac{\Phi, \varphi \vdash \psi}{\Phi \vdash \varphi \Rightarrow \psi} \text{ (}\Rightarrow\text{I)}$
$\frac{\Phi \vdash \varphi \& \psi}{\Phi \vdash \varphi} \text{ (&E}_1\text{)}$	$\frac{\Phi \vdash \varphi \& \psi}{\Phi \vdash \psi} \text{ (&E}_2\text{)}$	$\frac{\Phi \vdash \varphi \Rightarrow \psi \quad \Phi \vdash \varphi}{\Phi \vdash \psi} \text{ (}\Rightarrow\text{E)}$

FACT: if an IPL sequent $\Phi \vdash \phi$ is provable from the rules, it is provable without using the (CUT) rule. (This is called the **cut elimination theorem** for IPL.)

Proof theory

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$\frac{}{\Phi \vdash \text{true}} \text{ (TRUE)}$	$\frac{\Phi \vdash \varphi \quad \Phi \vdash \psi}{\Phi \vdash \varphi \& \psi} \text{ (&I)}$	$\frac{\Phi, \varphi \vdash \psi}{\Phi \vdash \varphi \Rightarrow \psi} \text{ (}\Rightarrow\text{I)}$
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FACT: if an IPL sequent $\Phi \vdash \phi$ is provable from the rules, it is provable without using the (CUT) rule. (This is called the **cut elimination theorem** for IPL.)

Simply-Typed Lambda Calculus provides a language for describing proofs in IPL and their properties...

Simply-Typed Lambda Calculus (STLC)

Types: $A, B, C, \dots ::=$

$G, G', G'' \dots$ “ground” types

unit unit type

$A \times B$ product type

$A \rightarrow B$ function type

Simply-Typed Lambda Calculus (STLC)

Types: $A, B, C, \dots ::=$

$G, G', G'' \dots$ “ground” types
unit unit type
 $A \times B$ product type
 $A \rightarrow B$ function type

Terms: $s, t, r, \dots ::=$

c^A constants (of given type A)
 x variable (countably many)
 $()$ unit value
 (s, t) pair
 $\text{fst } t \quad \text{snd } t$ projections
 $\lambda x : A. t$ function abstraction
 $s t$ function application

STLC

Some examples of terms:

▶ $\lambda z : (A \rightarrow B) \times (A \rightarrow C). \lambda x : A. ((\text{fst } z) x, (\text{snd } z) x)$

(has type $((A \rightarrow B) \times (A \rightarrow C)) \rightarrow (A \rightarrow (B \times C))$)

▶ $\lambda z : A \rightarrow (B \times C). (\lambda x : A. \text{fst}(z x), \lambda y : A. \text{snd}(z y))$

(has type $(A \rightarrow (B \times C)) \rightarrow ((A \rightarrow B) \times (A \rightarrow C))$)

▶ $\lambda z : A \rightarrow (B \times C). \lambda x : A. ((\text{fst } z) x, (\text{snd } z) x)$

(has no type)

STLC typing relation, $\Gamma \vdash t : A$

Γ ranges over **typing environments**

$$\Gamma ::= \diamond \mid \Gamma, x : A$$

(so typing environments are comma-separated snoc-lists of (variable,type)-pairs
— in fact only the lists whose variables are mutually distinct get used)

The typing relation $\Gamma \vdash t : A$ is inductively defined by the following rules, which make use of the following notation

$\Gamma \text{ ok}$ means: no variable occurs more than once in Γ

$\text{dom } \Gamma$ = finite set of variables occurring in Γ

STLC typing relation, $\Gamma \vdash t : A$

Typing rules for variables

$$\frac{\Gamma \text{ ok} \quad x \notin \text{dom } \Gamma}{\Gamma, x : A \vdash x : A} \text{ (VAR)}$$

$$\frac{\Gamma \vdash x : A \quad x' \notin \text{dom } \Gamma}{\Gamma, x' : A' \vdash x : A} \text{ (VAR')}$$

Typing rules for constants and unit value

$$\frac{\Gamma \text{ ok}}{\Gamma \vdash c^A : A} \text{ (CONS)}$$

$$\frac{\Gamma \text{ ok}}{\Gamma \vdash () : \text{unit}} \text{ (UNIT)}$$

STLC typing relation, $\Gamma \vdash t : A$

Typing rules for pairs and projections

$$\frac{\Gamma \vdash s : A \quad \Gamma \vdash t : B}{\Gamma \vdash (s, t) : A \times B} \text{ (PAIR)}$$

$$\frac{\Gamma \vdash t : A \times B}{\Gamma \vdash \text{fst } t : A} \text{ (FST)}$$

$$\frac{\Gamma \vdash t : A \times B}{\Gamma \vdash \text{snd } t : B} \text{ (SND)}$$

STLC typing relation, $\Gamma \vdash t : A$

Typing rules for function abstraction & application

$$\frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x : A. t : A \rightarrow B} \text{ (FUN)}$$

$$\frac{\Gamma \vdash s : A \rightarrow B \quad \Gamma \vdash t : A}{\Gamma \vdash st : B} \text{ (APP)}$$

Semantics of STLC types in a ccc

Given a cartesian closed category \mathbf{C} ,

any function M mapping ground types G to objects $M(G) \in \mathbf{C}$

extends to function $A \mapsto M[A] \in \mathbf{C}$ and $\Gamma \mapsto M[\Gamma] \in \mathbf{C}$ from STLC types and typing environments to \mathbf{C} -objects, by recursion on the structure of A :

$$M[G] = M(G)$$

$$M[\text{unit}] = 1$$

terminal object in \mathbf{C}

$$M[A \times B] = M[A] \times M[B]$$

product in \mathbf{C}

$$M[A \rightarrow B] = M[A] \rightarrow M[B]$$

exponential in \mathbf{C}

$$M[\diamond] = 1$$

terminal object in \mathbf{C}

$$M[\Gamma, x : A] = M[\Gamma] \times M[A]$$

product in \mathbf{C}

Semantics of STLC terms in a ccc

Given a cartesian closed category \mathbf{C} ,

given any function M mapping

- ▶ ground types G to \mathbf{C} -objects $M(G)$
(which extends to a function mapping all types to objects, $A \mapsto M[[A]]$,
as we have seen)

Semantics of STLC terms in a ccc

Given a cartesian closed category \mathbf{C} ,

given any function M mapping

- ▶ ground types G to \mathbf{C} -objects $M(G)$
- ▶ constants c^A to \mathbf{C} -morphisms $M(c^A) : 1 \rightarrow M[A]$
(In a category with a terminal object 1 , given an object $X \in \mathbf{C}$, morphisms $1 \rightarrow X$ are sometimes called **global elements** of X .)

Semantics of STLC terms in a ccc

Given a cartesian closed category \mathbf{C} ,

given any function M mapping

- ▶ ground types G to \mathbf{C} -objects $M(G)$
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we get a function mapping provable instances of the typing relation $\Gamma \vdash t : A$ to \mathbf{C} -morphisms

$$M[\Gamma \vdash t : A] : M[\Gamma] \rightarrow M[A]$$

defined by recursing over the proof of $\Gamma \vdash t : A$ from the typing rules (which follows the structure of t):

Semantics of STLC terms in a ccc

Variables:

$$M[\Gamma, x : A \vdash x : A] = M[\Gamma] \times M[A] \xrightarrow{\pi_2} M[A]$$

$$M[\Gamma, x' : A' \vdash x : A] =$$

$$M[\Gamma] \times M[A'] \xrightarrow{\pi_1} M[\Gamma] \xrightarrow{M[\Gamma \vdash x : A]} M[A]$$

Constants:

$$M[\Gamma \vdash c^A : A] = M[\Gamma] \xrightarrow{\langle \rangle} 1 \xrightarrow{M(c^A)} M[A]$$

Unit value:

$$M[\Gamma \vdash () : \text{unit}] = M[\Gamma] \xrightarrow{\langle \rangle} 1$$