Lecture 6

ccc

Recall:

Definition. $\mathbf{C}$ is a cartesian closed category (ccc) if it is a category with a terminal object, binary products and exponentials of any pair of objects.

Non-example of a ccc

The category **Mon** of monoids has a terminal object and binary products, but is <u>not</u> a ccc

because of the following bijections between sets, where 1 denotes a one-element set and the corresponding one-element monoid:

$$
\begin{aligned}
\mathbf{Set}(1, \mathtt{List}\,1) &\cong \mathbf{Mon}(\mathtt{List}\,1, \mathtt{List}\,1) \\
&\cong \mathbf{Mon}(1 \times \mathtt{List}\,1, \mathtt{List}\,1)
\end{aligned}
$$

by universal property of the free monoid $\mathtt{List}\,1$ on the one-element set 1

by Ex.Sh. 2, qu. 2 (1 is terminal in **Mon**)

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Since $\mathbf{Set}(1, \mathtt{List}\,1)$ is countably infinite, so is $\mathbf{Mon}(1 \times \mathtt{List}\,1, \mathtt{List}\,1)$.

Since the one-element monoid is terminal in **Mon**, for any $M \in \mathbf{Mon}$ we have that $\mathbf{Mon}(1, M)$ has just one element and hence

$$\mathbf{Mon}(1 \times \mathtt{List}\,1, \mathtt{List}\,1) \ncong \mathbf{Mon}(1, M)$$

Therefore no M can be the exponential of the objects $\mathtt{List}\,1$ and $\mathtt{List}\,1$ in **Mon**.

Cartesian closed pre-order

Recall that each pre-ordered set $(P, \sqsubseteq)$ gives a category $\mathbf{C}_P$. It is a ccc iff P has

- a greatest element $\top$: $\forall p \in P,\ p \sqsubseteq \top$
- binary meets $p \wedge q$:
  $\forall r \in P,\ r \sqsubseteq p \wedge q \;\Leftrightarrow\; r \sqsubseteq p \wedge r \sqsubseteq q$
- Heyting implications $p \to q$:
  $\forall r \in P,\ r \sqsubseteq p \to q \;\Leftrightarrow\; r \wedge p \sqsubseteq q$

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E.g. any Boolean algebra (with $p \to q = \neg p \vee q$).

E.g. $([0,1], \leq)$ with $\top = 1$, $p \wedge q = \min\{p, q\}$ and $p \to q = \begin{cases} 1 & \text{if } p \leq q \\ q & \text{if } q < p \end{cases}$

Intuitionistic Propositional Logic (IPL)

We present it in "natural deduction" style and only consider the fragment with conjunction and implication, with the following syntax:

Formulas of IPL: $\varphi, \psi, \theta, \ldots ::=$

$p, q, r, \ldots$	propositional identifiers
$\mathtt{true}$	truth
$\varphi \mathbin{\&} \psi$	conjunction
$\varphi \Rightarrow \psi$	implication

Sequents of IPL: $\Phi \;::=\; \diamond$ empty

Φ, ϕ non=empty

(so sequents are finite snoc-lists of formulas)

IPL entailment $\Phi \vdash \varphi$

The intended meaning of $\Phi \vdash \varphi$ is "the conjunction of the formulas in Φ implies the formula φ". The relation $\_ \vdash \_$ is inductively generated by the following rules:

$$\frac{}{\Phi, \varphi \vdash \varphi}\ (\mathrm{AX}) \qquad \frac{\Phi \vdash \varphi}{\Phi, \psi \vdash \varphi}\ (\mathrm{WK}) \qquad \frac{\Phi \vdash \varphi \qquad \Phi, \varphi \vdash \psi}{\Phi \vdash \psi}\ (\mathrm{CUT})$$

$$\frac{}{\Phi \vdash \mathtt{true}}\ (\mathrm{TRUE}) \qquad \frac{\Phi \vdash \varphi \qquad \Phi \vdash \psi}{\Phi \vdash \varphi \mathbin{\&} \psi}\ (\&\mathrm{I}) \qquad \frac{\Phi, \varphi \vdash \psi}{\Phi \vdash \varphi \Rightarrow \psi}\ ({\Rightarrow}\mathrm{I})$$

$$\frac{\Phi \vdash \varphi \mathbin{\&} \psi}{\Phi \vdash \varphi}\ (\&\mathrm{E}_1) \qquad \frac{\Phi \vdash \varphi \mathbin{\&} \psi}{\Phi \vdash \psi}\ (\&\mathrm{E}_2) \qquad \frac{\Phi \vdash \varphi \Rightarrow \psi \qquad \Phi \vdash \varphi}{\Phi \vdash \psi}\ ({\Rightarrow}\mathrm{E})$$

For example, if $\Phi = \diamond, \varphi \Rightarrow \psi, \psi \Rightarrow \theta$, then $\Phi \vdash \varphi \Rightarrow \theta$ is provable in IPL, because:

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{(AX)}
\UnaryInfC{$\Phi \vdash \psi \Rightarrow \theta$}
\RightLabel{(WK)}
\UnaryInfC{$\Phi, \varphi \vdash \psi \Rightarrow \theta$}
\AxiomC{}
\RightLabel{(AX)}
\UnaryInfC{$\diamond, \varphi \Rightarrow \psi \vdash \varphi \Rightarrow \psi$}
\RightLabel{(WK)}
\UnaryInfC{$\Phi \vdash \varphi \Rightarrow \psi$}
\RightLabel{(WK)}
\UnaryInfC{$\Phi, \varphi \vdash \varphi \Rightarrow \psi$}
\AxiomC{}
\RightLabel{(AX)}
\UnaryInfC{$\Phi, \varphi \vdash \varphi$}
\RightLabel{($\Rightarrow$E)}
\BinaryInfC{$\Phi, \varphi \vdash \psi$}
\RightLabel{($\Rightarrow$E)}
\BinaryInfC{$\Phi, \varphi \vdash \theta$}
\RightLabel{($\Rightarrow$I)}
\UnaryInfC{$\Phi \vdash \varphi \Rightarrow \theta$}
\end{prooftree}
$$

Semantics of IPL

in a cartesian closed pre-oder $(P, \sqsubseteq)$

Given a function M assigning a meaning to each propositional identifier p as an element $M(p) \in P$, we can assign meanings to IPL formula φ and sequents Φ as element $M[\![\varphi]\!], M[\![\Phi]\!] \in P$ by recursion on their structure:

$$
\begin{array}{rcll}
M[\![p]\!] & = & M(p) & \\
M[\![\mathtt{true}]\!] & = & \top & \text{greatest element} \\
M[\![\varphi \,\&\, \psi]\!] & = & M[\![\varphi]\!] \wedge M[\![\psi]\!] & \text{binary meet} \\
M[\![\varphi \Rightarrow \psi]\!] & = & M[\![\varphi]\!] \rightarrow M[\![\psi]\!] & \text{Heyting implication} \\
M[\![\diamond]\!] & = & \top & \text{greatest element} \\
M[\![\Phi, \varphi]\!] & = & M[\![\Phi]\!] \wedge M[\![\varphi]\!] & \text{binary meet}
\end{array}
$$

Semantics of IPL

in a cartesian closed pre-oder $(P, \sqsubseteq)$

Soundness Theorem. If $\Phi \vdash \varphi$ is provable from the rules of IPL, then $M[\![\Phi]\!] \sqsubseteq M[\![\varphi]\!]$ holds in any cartesian closed pre-order.

Proof. exercise (show that $\{(\Phi, \varphi) \mid M[\![\Phi]\!] \sqsubseteq M[\![\varphi]\!]\}$ is closed under the rules defining IPL entailment and hence contains $\{(\Phi, \varphi) \mid \Phi \vdash \varphi\}$)

Example

Peirce's Law $\diamond \vdash ((\varphi \Rightarrow \psi) \Rightarrow \varphi) \Rightarrow \varphi$

is not provable in IPL.

(whereas the formula $((\varphi \Rightarrow \psi) \Rightarrow \varphi) \Rightarrow \varphi$ is a classical tautology)

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For if $\diamond \vdash ((\varphi \Rightarrow \psi) \Rightarrow \varphi) \Rightarrow \varphi$ were provable in IPL, then by the Soundness Theorem we would have
$\top = M[\![\diamond]\!] \sqsubseteq M[\![((\varphi \Rightarrow \psi) \Rightarrow \varphi) \Rightarrow \varphi]\!]$.

But in the cartesian closed partial order $([0,1], \leq)$, taking $M(p) = 1/2$ and $M(q) = 0$, we get

$$
\begin{aligned}
M[\![((p \Rightarrow q) \Rightarrow p) \Rightarrow p]\!] &= ((1/2 \to 0) \to 1/2) \to 1/2 \\
&= (0 \to 1/2) \to 1/2 \\
&= 1 \to 1/2 \\
&= 1/2 \\
&\not\geq 1
\end{aligned}
$$

Semantics of IPL

in a cartesian closed pre-oder $(P, \sqsubseteq)$

Completeness Theorem. Given Φ, φ, if for all cartesian closed pre-orders $(P, \sqsubseteq)$ and all interpretations M of the propositional identifiers as elements of P, it is the case that $M[\![\Phi]\!] \sqsubseteq M[\![\varphi]\!]$ holds in P, then $\Phi \vdash \varphi$ is provable in IPL.

Semantics of IPL

in a cartesian closed pre-oder $(P, \sqsubseteq)$

Completeness Theorem. Given Φ, φ, if for all cartesian closed pre-orders $(P, \sqsubseteq)$ and all interpretations M of the propositional identifiers as elements of P, it is the case that $M[\![\Phi]\!] \sqsubseteq M[\![\varphi]\!]$ holds in P, then $\Phi \vdash \varphi$ is provable in IPL.

Proof. Define

$$\begin{aligned} P &\triangleq \{\text{formulas of IPL}\} \\ \varphi \sqsubseteq \psi &\triangleq \diamond, \varphi \vdash \psi \text{ is provable in IPL} \end{aligned}$$

Then one can show that $(P, \sqsubseteq)$ is a cartesian closed pre-ordered set. For this $(P, \sqsubseteq)$, taking M to be $M(p) = p$, one can show that $M[\![\Phi]\!] \sqsubseteq M[\![\varphi]\!]$ holds in P iff $\Phi \vdash \varphi$ is provable in IPL. $\square$