Lecture 6
Recall:

**Definition.** $\mathbf{C}$ is a **cartesian closed category (ccc)** if it is a category with a terminal object, binary products and exponentials of any pair of objects.
Non-example of a ccc

The category $\textbf{Mon}$ of monoids has a terminal object and binary products, but is not a ccc

because of the following bijections between sets, where $1$ denotes a one-element set and the corresponding one-element monoid:

$$\text{Set}(1, \text{List}
\text{1}) \cong \textbf{Mon}(\text{List}
\text{1}, \text{List}
\text{1}) \cong \textbf{Mon}(1 \times \text{List}
\text{1}, \text{List}
\text{1})$$

by universal property of the free monoid $\text{List}
\text{1}$ on the one-element set $1$

by Ex.Sh. 2, qu. 2
(1 is terminal in $\textbf{Mon}$)
Non-example of a ccc

The category \textbf{Mon} of monoids has a terminal object and binary products, but is \textbf{not} a ccc because of the following bijections between sets, where \textbf{1} denotes a one-element set and the corresponding one-element monoid:

\[
\text{Set}(\textbf{1}, \text{List}_1) \cong \text{Mon}(\text{List}_1, \text{List}_1) \\
\cong \text{Mon}(\textbf{1} \times \text{List}_1, \text{List}_1)
\]

Since \text{Set}(\textbf{1}, \text{List}_1) is countably infinite, so is \text{Mon}(\textbf{1} \times \text{List}_1, \text{List}_1).

Since the one-element monoid is terminal in \text{Mon}, for any \textbf{M} \in \text{Mon} we have that \text{Mon}(\textbf{1}, \textbf{M}) has just one element and hence

\[
\text{Mon}(\textbf{1} \times \text{List}_1, \text{List}_1) \ncong \text{Mon}(\textbf{1}, \textbf{M})
\]

Therefore no \textbf{M} can be the exponential of the objects \text{List}_1 and \text{List}_1 in \text{Mon}. 
Cartesian closed pre-order

Recall that each pre-ordered set \((P, \sqsubseteq)\) gives a category \(\mathcal{C}_P\). It is a ccc iff \(P\) has

- a greatest element \(\top\): \(\forall p \in P, p \sqsubseteq \top\)
- binary meets \(p \land q\):
  \(\forall r \in P, r \sqsubseteq p \land q \iff r \sqsubseteq p \land r \sqsubseteq q\)
- Heyting implications \(p \rightarrow q\):
  \(\forall r \in P, r \sqsubseteq p \rightarrow q \iff r \land p \sqsubseteq q\)
Cartesian closed pre-order

Recall that each pre-ordered set \((P, \sqsubseteq)\) gives a category \(\mathcal{C}_P\). It is a ccc iff \(P\) has

- a greatest element \(\top\): \(\forall p \in P, p \sqsubseteq \top\)
- binary meets \(p \land q\):
  \(\forall r \in P, r \sqsubseteq p \land q \iff r \sqsubseteq p \land r \sqsubseteq q\)
- Heyting implications \(p \to q\):
  \(\forall r \in P, r \sqsubseteq p \to q \iff r \land p \sqsubseteq q\)

E.g. any Boolean algebra (with \(p \to q = \neg p \lor q\)).

E.g. \(([0, 1], \leq)\) with \(\top = 1\), \(p \land q = \min\{p, q\}\) and \(p \rightarrow q = \begin{cases} 
1 & \text{if } p \leq q \\
q & \text{if } q < p
\end{cases}\)
Intuitionistic Propositional Logic (IPL)

We present it in “natural deduction” style and only consider the fragment with conjunction and implication, with the following syntax:

**Formulas** of IPL: $\varphi, \psi, \theta, \ldots ::=$
- $p, q, r, \ldots$ propositional identifiers
- $\text{true}$ truth
- $\varphi \& \psi$ conjunction
- $\varphi \Rightarrow \psi$ implication

**Sequents** of IPL: $\Phi ::=$ $\emptyset$ empty
- $\Phi, \varphi$ non-empty

(so sequents are finite snoc-lists of formulas)
The intended meaning of $\Phi \vdash \varphi$ is “the conjunction of the formulas in $\Phi$ implies the formula $\varphi$”. The relation $\_ \vdash \_\_ \_ \_ \_ \_ \_$ is inductively generated by the following rules:

- **(AX)**: $\Phi, \varphi \vdash \varphi$
- **(WK)**: $\Phi \vdash \varphi$
- **(CUT)**: $\Phi \vdash \varphi$, $\varphi \vdash \psi$)
- **(TRUE)**: $\Phi \vdash \text{true}$
- **(&I)**: $\Phi \vdash \varphi$, $\Phi \vdash \psi$)
- **(⇒I)**: $\Phi, \varphi \vdash \psi$
- **(&E₁)**: $\Phi \vdash \varphi \& \psi$
- **(&E₂)**: $\Phi \vdash \varphi \& \psi$
- **(⇒E)**: $\Phi \vdash \varphi \Rightarrow \psi$, $\Phi \vdash \varphi$
For example, if $\Phi = \diamond, \varphi \Rightarrow \psi, \psi \Rightarrow \theta$, then $\Phi \vdash \varphi \Rightarrow \theta$ is provable in IPL, because:

$$
\Phi \vdash \psi \Rightarrow \theta \quad (\text{AX}) \\
\Phi, \varphi \vdash \psi \Rightarrow \theta \quad (\text{WK})
$$

$$
\diamond, \varphi \Rightarrow \psi \vdash \varphi \Rightarrow \psi \quad (\text{AX}) \\
\Phi \vdash \varphi \Rightarrow \psi \quad (\text{WK}) \\
\Phi, \varphi \vdash \varphi \Rightarrow \psi \quad (\text{WK}) \\
\Phi, \varphi \vdash \psi \quad (\Rightarrow \text{E}) \\
\Phi, \varphi \vdash \theta \quad (\Rightarrow \text{E}) \\
\Phi \vdash \varphi \Rightarrow \theta \quad (\Rightarrow \text{I})
$$
Semantics of IPL
in a cartesian closed pre-order \((P, \sqsubseteq)\)

Given a function \(M\) assigning a meaning to each propositional identifier \(p\) as an element \(M(p) \in P\), we can assign meanings to IPL formula \(\varphi\) and sequents \(\Phi\) as element \(M[\varphi], M[\Phi] \in P\) by recursion on their structure:

\[
\begin{align*}
M[p] &= M(p) \\
M[\text{true}] &= \top & \text{greatest element} \\
M[\varphi \& \psi] &= M[\varphi] \land M[\psi] & \text{binary meet} \\
M[\varphi \Rightarrow \psi] &= M[\varphi] \rightarrow M[\psi] & \text{Heyting implication} \\
M[\Diamond] &= \top & \text{greatest element} \\
M[\Phi, \varphi] &= M[\Phi] \land M[\varphi] & \text{binary meet}
\end{align*}
\]
Semantics of IPL  
in a cartesian closed pre-order \((P, \sqsubseteq)\)

**Soundness Theorem.** If \(\Phi \vdash \varphi\) is provable from the rules of IPL, then \(M[\Phi] \sqsubseteq M[\varphi]\) holds in any cartesian closed pre-order.

**Proof.** Exercise (show that \(\{(\Phi, \varphi) \mid M[\Phi] \sqsubseteq M[\varphi]\}\) is closed under the rules defining IPL entailment and hence contains \(\{(\Phi, \varphi) \mid \Phi \vdash \varphi\}\))
Example

Peirce’s Law \( \lozenge \vdash ((\varphi \Rightarrow \psi) \Rightarrow \varphi) \Rightarrow \varphi \)
is not provable in IPL.

(whereas the formula \( ((\varphi \Rightarrow \psi) \Rightarrow \varphi) \Rightarrow \varphi \) is a classical tautology)
Example

Peirce’s Law $\lozenge \vdash ((\varphi \Rightarrow \psi) \Rightarrow \varphi) \Rightarrow \varphi$
is not provable in IPL.

(Whereas the formula $((\varphi \Rightarrow \psi) \Rightarrow \varphi) \Rightarrow \varphi$ is a classical tautology)

For if $\lozenge \vdash ((\varphi \Rightarrow \psi) \Rightarrow \varphi) \Rightarrow \varphi$ were provable in IPL, then by the
Soundness Theorem we would have
$\top = M[\lozenge] \subseteq M[(\varphi \Rightarrow \psi) \Rightarrow \varphi].$

But in the cartesian closed partial order $([0, 1], \leq)$, taking $M(p) = \frac{1}{2}$ and $M(q) = 0$, we get

$M[((p \Rightarrow q) \Rightarrow p) \Rightarrow p] = (\frac{1}{2} \rightarrow 0) \rightarrow \frac{1}{2} \rightarrow \frac{1}{2} = (0 \rightarrow \frac{1}{2}) \rightarrow \frac{1}{2} = 1 \rightarrow \frac{1}{2} = \frac{1}{2} \not\geq 1$
Semantics of IPL
in a cartesian closed pre-order \((P, \sqsubseteq)\)

**Completeness Theorem.** Given \(\Phi, \varphi\), if for all cartesian closed pre-orders \((P, \sqsubseteq)\) and all interpretations \(M\) of the propositional identifiers as elements of \(P\), it is the case that \(M[\Phi] \sqsubseteq M[\varphi]\) holds in \(P\), then \(\Phi \vdash \varphi\) is provable in IPL.
Semantics of IPL
in a cartesian closed pre-oder \((P, \sqsubseteq)\)

Completeness Theorem. Given \(\Phi, \varphi\), if for all cartesian closed pre-orders \((P, \sqsubseteq)\) and all interpretations \(M\) of the propositional identifiers as elements of \(P\), it is the case that \(M[\Phi] \sqsubseteq M[\varphi]\) holds in \(P\), then \(\Phi \vdash \varphi\) is provable in IPL.

Proof. Define

\[
P \triangleq \{ \text{formulas of IPL} \}
\]

\[
\varphi \sqsubseteq \psi \triangleq \Diamond, \varphi \vdash \psi \text{ is provable in IPL}
\]

Then one can show that \((P, \sqsubseteq)\) is a cartesian closed pre-ordered set. For this \((P, \sqsubseteq)\), taking \(M\) to be \(M(p) = p\), one can show that \(M[\Phi] \sqsubseteq M[\varphi]\) holds in \(P\) iff \(\Phi \vdash \varphi\) is provable in IPL. \qed