Category Theory

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Module L108, Part III and MPhil. ACS
2020 Computer Science Tripos
University of Cambridge

Course web page

Go to

http://www.cl.cam.ac.uk/teaching/1920/L108

for

- these slides
- exercise sheets
- office hours : Mondays 2-3pm (FC08)
- pointers to some additional material

Recommended text for the course is:

[Awodey] Steve Awodey, *Category theory*, Oxford University Press (2nd ed.), 2010.

Assessment

- **A graded exercise sheet** (25% of the final mark).
  Exercise Sheet 4, issued in lecture 10 on Tuesday 10 November 2019, with solutions due in at the Graduate Office Graduate Office (FS03) by 16:00 on Tuesday 19 November 2019.
- **A take-home test** (75% of the final mark).
  The take-home test will be issued on Thursday 16 January 2020 at 16:00. Solutions are due in at the Graduate Office (FS03) by 16:00 on Monday 20 January 2020.

Lecture 1

What is category theory?

> What we are probably seeking is a "purer" view of functions: a theory of functions in themselves, not a theory of functions derived from sets. What, then, is a pure theory of functions? Answer: category theory.
>
> Dana Scott, *Relating theories of the λ-calculus*, p406

set theory gives an "element-oriented" account of mathematical structure, whereas

category theory takes a 'function-oriented" view – understand structures not via their elements, but by how they transform, i.e. via morphisms.

(Both theories are part of Logic, broadly construed.)

GENERAL THEORY OF NATURAL EQUIVALENCES

BY

SAMUEL EILENBERG AND SAUNDERS MACLANE

CONTENTS

Introduction. The subject matter of this paper is best explained by an example, such as that of the relation between a vector space L and its "dual"

Presented to the Society, September 8, 1942; received by the editors May 15, 1945.

Category Theory emerges

1945 Eilenberg† and MacLane†
General Theory of Natural Equivalences,
Trans AMS 58, 231–294

(algebraic topology, abstract algebra)

1950s Grothendieck† (algebraic geometry)

1960s Lawvere (logic and foundations)

1970s Joyal and Tierney† (elementary topos theory)

1980s Dana Scott, Plotkin

(semantics of programming languages)

Lambek† (linguistics)

Category Theory and Computer Science

"Category theory has. . . become part of the standard "tool-box" in many areas of theoretical informatics, from programming languages to automata, from process calculi to Type Theory."

Dagstuhl Perpectives Workshop on *Categorical Methods at the Crossroads*
April 2014

This course

basic concepts of category theory

$$
\begin{tikzcd}
\textbf{adjunction} & \textbf{natural transformation} \arrow[l] \\
\textbf{category} \arrow[r] & \textbf{functor} \arrow[u]
\end{tikzcd}
$$

applied to $\left\{ \begin{array}{l} \textbf{propositional logic} \\ \textbf{typed lambda-calculus} \\ \textbf{functional programming} \end{array} \right.$

Definition

A category $\mathbf{C}$ is specified by

- a set $\boxed{\mathtt{obj}\,\mathbf{C}}$ whose elements are called $\mathbf{C}$-objects
- for each $X, Y \in \mathtt{obj}\,\mathbf{C}$, a set $\boxed{\mathbf{C}(X,Y)}$ whose elements are called $\mathbf{C}$-morphisms from X to Y

(so far, that is just what some people call a **directed graph**)

Definition

A category $\mathbf{C}$ is specified by

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- for each $X, Y \in \mathtt{obj}\,\mathbf{C}$, a set $\boxed{\mathbf{C}(X, Y)}$ whose elements are called $\mathbf{C}$-morphisms from X to Y
- a function assigning to each $X \in \mathtt{obj}\,\mathbf{C}$ an element $\boxed{\mathrm{id}_X \in \mathbf{C}(X, X)}$ called the identity morphism for the $\mathbf{C}$-object X
- a function assigning to each $f \in \mathbf{C}(X, Y)$ and $g \in \mathbf{C}(Y, Z)$ (where $X, Y, Z \in \mathtt{obj}\,\mathbf{C}$) an element $\boxed{g \circ f \in \mathbf{C}(X, Z)}$ called the composition of $\mathbf{C}$-morphisms f and g and satisfying. . .

Definition, continued

satisfying. . .

- associativity: for all $X, Y, Z, W \in \mathtt{obj}\,\mathbf{C}$, $f \in \mathbf{C}(X,Y)$, $g \in \mathbf{C}(Y,Z)$ and $h \in \mathbf{C}(Z,W)$

$$\boxed{h \circ (g \circ f) = (h \circ g) \circ f}$$

- unity: for all $X, Y \in \mathtt{obj}\,\mathbf{C}$ and $f \in \mathbf{C}(X,Y)$

$$\boxed{\mathtt{id}_Y \circ f = f = f \circ \mathtt{id}_X}$$

Example: category of sets, **Set**

- `obj` **Set** $=$ some fixed universe of sets

  (more on universes later)

- **Set**$(X, Y) =$
  $\{f \subseteq X \times Y \mid f$ is single-valued and total$\}$

Cartesian product of sets X and Y is the set of all ordered pairs (x, y) with $x \in X$ and $y \in Y$.
Equality of ordered pairs:
$(x, y) = (x', y') \Leftrightarrow x = x' \wedge y = y'$

Example: category of sets, **Set**

- $\texttt{obj}\,\mathbf{Set}$ = some fixed universe of sets

  (more on universes later)

- $\mathbf{Set}(X, Y) =$
  $\{f \subseteq X \times Y \mid f$ is single-valued and total$\}$

$$\forall x \in X, \forall y, y' \in Y,\\ (x, y) \in f \wedge (x, y') \in f \Rightarrow y = y'$$

$$\forall x \in X, \exists y \in Y,\\ (x, y) \in f$$

Example: category of sets, **Set**

- $\mathtt{obj}\,\mathbf{Set}$ = some fixed universe of sets

  (more on universes later)

- $\mathbf{Set}(X, Y) =$
  $\{f \subseteq X \times Y \mid f \text{ is single-valued and total}\}$
- $\mathtt{id}_X = \{(x, x) \mid x \in X\}$
- composition of $f \in \mathbf{Set}(X, Y)$ and $g \in \mathbf{Set}(Y, Z)$ is

$$g \circ f = \{(x, z) \mid \exists y \in Y,\ (x, y) \in f \wedge (y, z) \in g\}$$

(check that associativity and unity properties hold)

Example: category of sets, **Set**

Notation. Given $f \in \mathbf{Set}(X, Y)$ and $x \in X$, it is usual to write $\boxed{f\,x}$ (or $f(x)$) for the unique $y \in Y$ with $(x, y) \in f$.

Thus

$$\mathtt{id}_X\, x = x$$
$$(g \circ f)\, x = g(f\, x)$$

Domain and codomain

Given a category $\mathbf{C}$,

write $\boxed{f : X \to Y}$ or $\boxed{X \xrightarrow{f} Y}$

to mean that $f \in \mathbf{C}(X, Y)$,

in which case one says

object X is the domain of the morphism f
object Y is the codomain of the morphism f

and writes

$$X = \mathrm{dom}\, f \qquad Y = \mathrm{cod}\, f$$

(Which category $\mathbf{C}$ we are referring to is left implicit with this notation.)

Commutative diagrams

in a category **C**:

a diagram is

> a directed graph whose vertices are **C**-objects and whose edges are **C**-morphisms

and the diagram is commutative (or commutes) if

> any two finite paths in the graph between any two vertices determine equal morphisms in the category under composition

Commutative diagrams

Examples:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "f"'] & Y \arrow[d, "\mathrm{id}_Y"] \arrow[dr, "g"] & \\ & Y \arrow[r, "g"'] & Z \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[r, "h \circ (g \circ f)"] \arrow[d, "f"'] & W \\ Y \arrow[ur, "h \circ g"] \arrow[r, "g"'] & Z \arrow[u, "h"'] \end{tikzcd}$$

Alternative notations

I will often just write

$\mathbf{C}$ for $\mathtt{obj}\,\mathbf{C}$
$\mathtt{id}$ for $\mathtt{id}_X$

Some people write

$\mathtt{Hom}_{\mathbf{C}}(X, Y)$ for $\mathbf{C}(X, Y)$
1_X for $\mathtt{id}_X$
$g\,f$ for $g \circ f$

I use "applicative order" for morphism composition; other people use "diagrammatic order" and write

$f;g$ (or $f\,g$) for $g \circ f$

Alternative definition of category

The definition given here is "dependent-type friendly".

See [Awodey, Definition 1.1] for an equivalent formulation:

> One gives the whole set of morphisms $\mathtt{mor}\,\mathbf{C}$
> (in bijection with $\sum_{X,Y \in \mathtt{obj}\,\mathbf{C}} \mathbf{C}(X,Y)$ in my definition)
> plus functions
> $$\mathtt{dom}, \mathtt{cod} : \mathtt{mor}\,\mathbf{C} \to \mathtt{obj}\,\mathbf{C}$$
> $$\mathtt{id} : \mathtt{obj}\,\mathbf{C} \to \mathtt{mor}\,\mathbf{C}$$
> and a *partial* function for composition
> $$\_\circ\_ : \mathtt{mor}\,\mathbf{C} \times \mathtt{mor}\,\mathbf{C} \rightharpoonup \mathtt{mor}\,\mathbf{C}$$
> defined at (f,g) iff $\mathtt{cod}\,f = \mathtt{dom}\,g$
> and satisfying the associativity and unity equations.