Category Theory

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Module L108, Part III and MPhil. ACS 2020 Computer Science Tripos University of Cambridge

LO 1

Course web page

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Go to http://www.cl.cam.ac.uk/teaching/1920/L108 for
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- these slides
- exercise sheets
- office hours : Mondays 2-3pm (FC08)
- pointers to some additional material

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Recommended text for the course is:

[Awodey] Steve Awodey, Category theory,
Oxford University Press (2nd ed.), 2010.
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LO 2

Assessment

- ► A graded exercise sheet (25% of the final mark). Exercise Sheet 4, issued in lecture 10 on Tuesday 10 November 2019, with solutions due in at the Graduate Office (FS03) by 16:00 on Tuesday 19 November 2019.
- ➤ A take-home test (75% of the final mark). The take-home test will be issued on Thursday 16 January 2020 at 16:00. Solutions are due in at the Graduate Office (FS03) by 16:00 on Monday 20 January 2020.

LU 3

Lecture 1

What is category theory?

What we are probably seeking is a "purer" view of functions: a theory of functions in themselves, not a theory of functions derived from sets. What, then, is a pure theory of functions? Answer: category theory.

Dana Scott, Relating theories of the λ -calculus, p406

set theory gives an "element-oriented" account of mathematical structure, whereas

category theory takes a 'function-oriented' view — understand structures not via their elements, but by how they transform, i.e. via morphisms.

(Both theories are part of Logic, broadly construed.)

GENERAL THEORY OF NATURAL EQUIVALENCES

SAMUEL EILENBERG AND SAUNDERS MACLANE

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Introduction. The subject matter of this paper is best explained by an example, such as that of the relation between a vector space L and its "dual"

Presented to the Society, September 8, 1942; received by the editors May 15, 1945.

Category Theory emerges

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1945 Eilenberg<sup>†</sup> and MacLane<sup>†</sup>
         General Theory of Natural Equivalences,
        Trans AMS 58, 231-294
         (algebraic topology, abstract algebra)
1950s Grothendieck<sup>†</sup> (algebraic geometry)
1960s Lawvere (logic and foundations)
1970s Joyal and Tierney<sup>†</sup> (elementary topos theory)
1980s Dana Scott, Plotkin
         (semantics of programming languages)
        Lambek<sup>†</sup> (linguistics)
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Category Theory and Computer Science

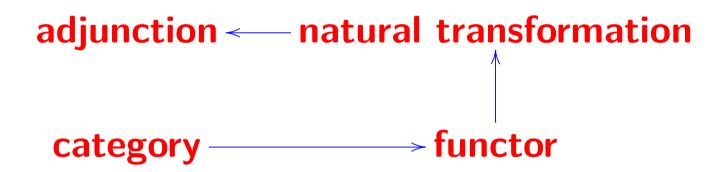
"Category theory has... become part of the standard "tool-box" in many areas of theoretical informatics, from programming languages to automata, from process calculi to Type Theory."

Dagstuhl Perpectives Workshop on Categorical Methods at the Crossroads

April 2014

This course

basic concepts of category theory



applied to
 propositional logic
 typed lambda-calculus
 functional programming

Definition

A category C is specified by

- ▶ a set obj C whose elements are called C-objects
- ▶ for each $X, Y \in \text{obj } \mathbb{C}$, a set $\boxed{\mathbb{C}(X, Y)}$ whose elements are called \mathbb{C} -morphisms from X to Y

(so far, that is just what some people call a directed graph)

Definition

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- ▶ a function assigning to each $X \in \text{obj } \mathbb{C}$ an element $id_X \in \mathbb{C}(X,X)$ called the identity morphism for the \mathbb{C} -object X
- ▶ a function assigning to each $f \in C(X,Y)$ and $g \in C(Y,Z)$ (where $X,Y,Z \in \text{obj } C$) an element $g \circ f \in C(X,Z)$ called the composition of C-morphisms f and g and satisfying...

Definition, continued

satisfying...

▶ associativity: for all $X, Y, Z, W \in \text{obj } \mathbb{C}$, $f \in \mathbb{C}(X, Y), g \in \mathbb{C}(Y, Z)$ and $h \in \mathbb{C}(Z, W)$

$$h\circ (g\circ f)=(h\circ g)\circ f$$

ightharpoonup unity: for all $X,Y\in {
m obj}\,{f C}$ and $f\in {f C}(X,Y)$

$$\operatorname{id}_Y \circ f = f = f \circ \operatorname{id}_X$$

- obj Set = some fixed universe of sets
 (more on universes later)
- Set $(X, Y) = \{ f \subseteq X \times Y \mid f \text{ is single-valued and total} \}$

Cartesian product of sets X and Y is the set of all ordered pairs (x, y) with $x \in X$ and $y \in Y$.

Equality of ordered pairs:

$$(x,y)=(x',y')\Leftrightarrow x=x'\wedge y=y'$$

- obj Set = some fixed universe of sets
 (more on universes later)
- ► $Set(X,Y) = \{ f \subseteq X \times Y \mid f \text{ is single-valued and total} \}$

```
\forall x \in X, \forall y, y' \in Y, \\ (x, y) \in f \land (x, y') \in f \Rightarrow y = y'
```

 $\forall x \in X, \exists y \in Y, \\ (x,y) \in f$

- obj Set = some fixed universe of sets
 (more on universes later)
- ► $Set(X,Y) = \{ f \subseteq X \times Y \mid f \text{ is single-valued and total} \}$
- $\blacktriangleright id_X = \{(x, x) \mid x \in X\}$
- ▶ composition of $f \in \mathbf{Set}(X,Y)$ and $g \in \mathbf{Set}(Y,Z)$ is

$$g \circ f = \{(x, z) \mid \exists y \in Y, (x, y) \in f \land (y, z) \in g\}$$

(check that associativity and unity properties hold)

Notation. Given $f \in \mathbf{Set}(X,Y)$ and $x \in X$, it is usual to write f(x) (or f(x)) for the unique $y \in Y$ with $(x,y) \in f$.

Thus

$$id_X x = x$$
$$(g \circ f) x = g(f x)$$

Domain and codomain

Given a category C,

write
$$f: X \to Y$$
 or $X \xrightarrow{f} Y$

to mean that $f \in C(X,Y)$,

in which case one says

object X is the domain of the morphism f object Y is the codomain of the morphism f

and writes

$$X = \operatorname{dom} f$$
 $Y = \operatorname{cod} f$

(Which category C we are referring to is left implicit with this notation.)

Commutative diagrams

in a category **C**:

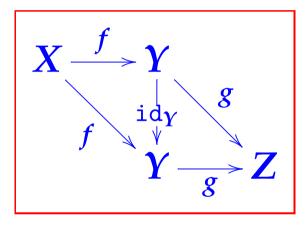
a diagram is

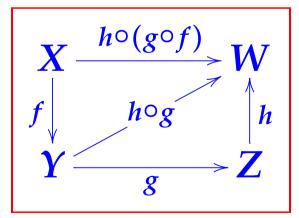
a directed graph whose vertices are C-objects and whose edges are C-morphisms

and the diagram is commutative (or commutes) if any two finite paths in the graph between any two vertices determine equal morphisms in the category under composition

Commutative diagrams

Examples:





Alternative notations

I will often just write

```
C for obj C id for id_X
```

Some people write

```
\operatorname{Hom}_{\mathbf{C}}(X,Y) for \mathbf{C}(X,Y)

1_X for \operatorname{id}_X

gf for g\circ f
```

I use "applicative order" for morphism composition; other people use "diagrammatic order" and write

$$f;g$$
 (or fg) for $g \circ f$

Alternative definition of category

The definition given here is "dependent-type friendly".

See [Awodey, Definition 1.1] for an equivalent formulation:

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One gives the whole set of morphisms \operatorname{mor} \mathbf{C} (in bijection with \sum_{X,Y\in\operatorname{obj}\mathbf{C}}\mathbf{C}(X,Y) in my definition) plus functions \operatorname{dom},\operatorname{cod}:\operatorname{mor}\mathbf{C}\to\operatorname{obj}\mathbf{C} \operatorname{id}:\operatorname{obj}\mathbf{C}\to\operatorname{mor}\mathbf{C} and a partial function for composition -\circ = \operatorname{mor}\mathbf{C}\times\operatorname{mor}\mathbf{C}\to\operatorname{mor}\mathbf{C} defined at (f,g) iff \operatorname{cod} f=\operatorname{dom} g and satisfying the associativity and unity equations.
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