2018/19 MPhil ACS / CST Part III  
*Category Theory (L108)*  
Exercise Sheet 4 [GRADED]  
Solutions due by 16:00 on Thursday 15 November

1. **[8/40 marks]** Let $C$ be a category with binary products. Given a $C$-object $X$, the *diagonal* morphism $\delta_X \in C(X, X \times X)$ and the *twist* morphism $\tau_X \in C(X \times X, X \times X)$ are defined by:

\[
\delta_X \triangleq \langle \text{id}_X, \text{id}_X \rangle \quad \tau_X \triangleq \langle \pi_2, \pi_1 \rangle
\]

(a) For all $C$-objects $X$ and $Y$ and all morphisms $f, g \in C(X, Y)$, show that

\[
t_Y \circ (g \times f) = (f \times g) \circ \tau_X
\]

(where $f \times g \in C(X \times X, Y \times Y)$ is the product of morphisms introduced in Ex. Sh. 2, question 1b).

(b) For each $f \in C(X, Y)$, show that $\delta_Y \circ f = (f \times f) \circ \delta_X$

(c) Show that $\tau_X \circ \delta_X = \delta_X$.

(d) Show that $\tau_X \circ \tau_X = \text{id}_{X \times X}$.

2. **[10/40 marks]** Let $C$ be a category. Given $C$-objects $X$ and $Y$ and morphisms $f, g \in C(X, Y)$, an *equalizer* for $f$ and $g$ is by definition a $C$-object $E$ and a morphism $m \in C(E, X)$ such that

- $f \circ m = g \circ m \in C(E, Y)$ and
- for all $C$-objects $Z$ and morphisms $h \in C(Z, X)$, if $f \circ h = g \circ h \in C(Z, Y)$, then there exists a unique morphism $k \in C(Z, E)$ satisfying $m \circ k = h$.

\[
\begin{array}{c}
Z \\
\downarrow^k \\
E \xrightarrow{m} X \xrightarrow{f} Y
\end{array}
\]

(a) Show that every equalizer is a monomorphism (see Ex. Sh. 1, question 4).

(b) Suppose that $f \in C(X, Y)$ is a *split monomorphism*, that is, there is a morphism $g \in C(Y, X)$ with $g \circ f = \text{id}_X$ (see Ex. Sh. 1, question 4). Show that $f : X \to Y$ is the equalizer of some pair of morphisms.

(c) Suppose the diagram

\[
\begin{array}{c}
U \xrightarrow{e} V \\
\downarrow^u \quad \downarrow^v \\
E \xrightarrow{m} X
\end{array}
\]

commutes in $C$ (i.e. $m \circ u = v \circ e$), that $m$ is an equalizer and that $e$ is an epimorphism (see Ex. Sh. 1, question 5). Show that there is a unique morphism $k \in C(V, E)$ such that $m \circ k = v$ and $k \circ e = u$. 
(d) Show that the category $\mathbf{Set}$ of sets and functions possesses equalizers for all parallel pairs of morphisms and that every monomorphism is an equalizer. Is every monomorphism in $\mathbf{Set}$ a split monomorphism?

3. [10/40 marks] Let $X$ be an object of a category $\mathbf{C}$. The slice category $\mathbf{C}/X$ is defined by:

- The objects of $\mathbf{C}/X$ are pairs $(A, p)$ where $A \in \text{obj} \mathbf{C}$ and $p \in \mathbf{C}(A, X)$.
- Given two such objects $(A, p)$ and $(B, q)$, a morphism $f : (A, p) \to (B, q)$ in $\mathbf{C}/X$ is a $\mathbf{C}$-morphism $f \in \mathbf{C}(A, B)$ such that $q \circ f = p$

$$
\begin{array}{c}
A \\
\downarrow f \\
B \\
\downarrow p \\
X \\
\end{array}
$$

- Composition and identities in $\mathbf{C}/X$ are given by those in $\mathbf{C}$.

(a) Show that $\mathbf{C}/X$ always has a terminal object.

(b) Show that if $\mathbf{C}$ has an initial object and binary coproducts, then so does $\mathbf{C}/X$.

(c) When $\mathbf{C} = \mathbf{Set}$, the category of sets and functions, show that $\mathbf{Set}/X$ has binary products. [Hint: given $(A, p), (B, q) \in \text{obj} \mathbf{Set}/X$, consider a suitable subset of $\{(a, b) \mid a \in A \land b \in B\}$.

4. [4/40 marks] Let $\mathbf{C} = \mathbf{Set}^{\text{op}}$ be the opposite category of the category $\mathbf{Set}$ of sets and functions.

(a) State, without proof, what is the product in $\mathbf{C}$ of two objects $X$ and $Y$.

(b) Show by example that there are objects $X$ and $Y$ in $\mathbf{C}$ for which there is no exponential and hence that $\mathbf{C}$ is not a cartesian closed category.

5. [8/40 marks] Let $\mathbf{C}$ be a cartesian closed category. Writing $X \to Y$ for the exponential $Y^X$ of two objects in $\mathbf{C}$, define $P(X, Y)$ to be the $\mathbf{C}$-object $((X \to Y) \to X) \to X$.

(a) By giving a suitable simply typed lambda calculus term in the internal language of $\mathbf{C}$, or otherwise, show that for any $\mathbf{C}$-object $X$, there is a morphism $p_X : 1 \to P(X, X)$ in $\mathbf{C}$.

(b) When $\mathbf{C} = \mathbf{Set}$, show that for any sets $X$ and $Y$ (including the case where one or other of them is empty), there is always some morphism $1 \to P(X, Y)$.

(c) Give an example of a cartesian closed category $\mathbf{C}$ containing objects $X$ and $Y$ for which there is no morphism $1 \to P(X, Y)$. [Hint: recall the example on page 63 in Lecture 6.]

(d) Call a term $t$ of the simply typed lambda calculus pure if it does not contain any constant symbols. Explain why part (c) implies that there is no pure term $t$ such that $\vdash t : ((G \to G') \to G) \to G$ holds, where $G$ and $G'$ are distinct ground types.